ENGINEERING MECHANICS I (STATICS)

Course Number: CEng 1041

by:- Mubarek Zeyne (MSc)

email: 2008mubarek@gmail.com

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Disclaimer

This lecture material is almost entirely taken from book called Engineering Mechanics by J.L. Meriam & L.G. Kraige, 7th edition

COURSE CONTENTS

1. Scalars and Vectors:
2. Force Systems
3. Equilibrium
4. Analysis of simple Structures
5. Internal Actions in beams
6. Centroids
7. Area Moments of Inertia
8. Friction

INTRODUCTION

What is mechanics and its application in Engineering science?

- Mechanics is the physical science which deals with the effects of forces on objects. It is divided into three parts: mechanics of rigid bodies, mechanics of deformable bodies, and mechanics of fluids.
- It is the oldest of the physical sciences. The early history of this subject is synonymous with the very beginnings of engineering.
- No other subject plays a greater role in engineering analysis than mechanics. Although the principles of mechanics are few, they have wide application in engineering. [1]

- The subject of mechanics is logically divided into two parts: statics, which concerns the equilibrium of bodies under action of forces, and dynamics, which concerns the motion of bodies. Engineering Mechanics is divided into these two parts, Vol. 1 Statics and Vol. 2 Dynamics

- Statics deals primarily with the calculation of external forces which act on rigid bodies in equilibrium. Determination of the internal deformations belongs to the study of the mechanics of deformable bodies or mechanics of materials.

Basic Concepts

- Space is the geometric region occupied by bodies whose positions are described by linear and angular measurements relative to a coordinate system. For 3D problems, three independent coordinates are needed. For 2D problems, only two coordinates are required.
- Time is the measure of the succession of events and is a basic quantity in dynamics. Time is not directly involved in the analysis of statics problems.
- Mass is a measure of the inertia of a body, which is its resistance to a change of velocity. Mass can also be thought of as the quantity of matter in a body.

- Force is the action of one body on another. A force tends to move a body in the direction of its action. The action of a force is characterized by its magnitude, by the direction of its action, and by its point of application. Thus force is a vector quantity, and its properties are discussed in detail in next session.
- A particle is a body of negligible dimensions. In the mathematical sense, a particle is a body whose dimensions are considered to be near zero so that we may analyze it as a mass concentrated at a point. We often choose a particle as a differential element of a body. We may treat a body as a particle when its dimensions are irrelevant to the description of its position or the action of forces applied to it.
- Rigid body. A body is considered rigid when the change in distance between any two of its points is negligible for the purpose at hand.

Fundamental Principles

Newton's Laws

Law I. A particle remains at rest or continues to move with uniform velocity (in a straight line with a constant speed) if there is no unbalanced force acting on it.

Law II. The acceleration of a particle is proportional to the vector sum of forces acting on it, and is in the direction of this vector sum.

Law III. The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and collinear (they lie on the same line).

♠ The law of gravitation is expressed by the equation

$$F = G\frac{m_1 m_2}{r^2}$$

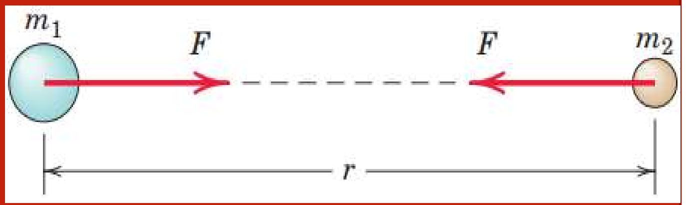
where

F= the mutual force of attraction between two particles

G = a universal constant known as the constant of gravitation, $6.673\text{x}10^{-11}\ m^3/(kg\ *s^2)$

m_1, m_2 = the masses of the two particles

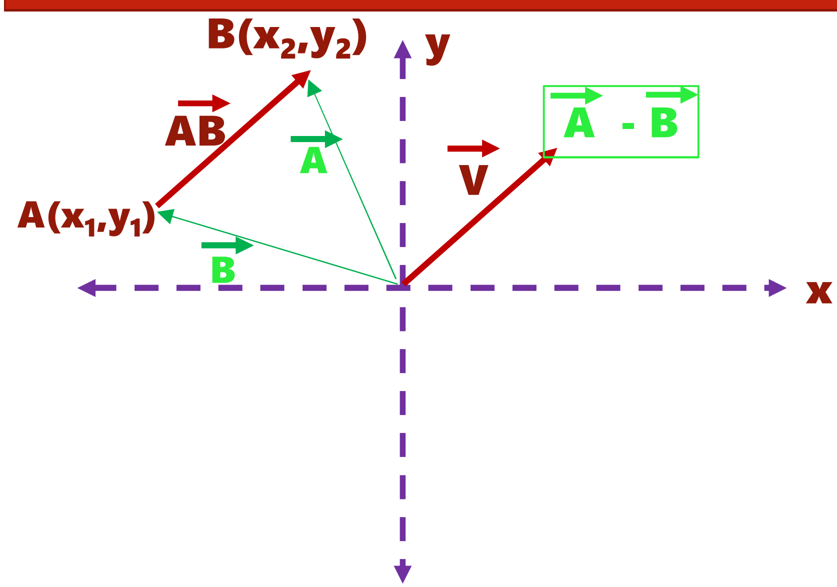
r =the distance between the centers of the particles



The mutual forces F obey the law of action and reaction, since they are equal and opposite and are directed along the line joining the centers of the particles.

SCALARS AND VECTORS

- We use two kinds of quantities in mechanics—scalars and vectors.
- ✓ Scalars are physical quantities that can be completely described (measured) by their magnitude alone. These quantities do not need a direction to point out their application (Just a value to quantify their measurability). They only need the magnitude and the unit of measurement to fully describe them. Examples of scalar quantities are time(s), area(m^2), volume(m^3), mass(kg),density(kg/m^3), speed(m/s), and energy(Wat)
- Vector quantities, on the other hand, possess direction as well as magnitude with its unit. E.g.:-displacement(m), velocity(m/s), acceleration(m/s^2), force(N, $kg.m/s^2$), moment(N.m), and momentum(N.s, kg.m/s).
- When writing vector equations, always be certain to preserve the mathematical distinction between vectors and scalars. In handwritten work, use a distinguishing mark for each vector quantity, such as an underline, $\underline{V}$, or an arrow over the symbol, $\vec{V}$, to take the place of boldface type in print.



Position vector is a vector that locates a given point in reference to origin(point of interest).

- Let $\vec{AB}$ is a vector with initial point (x_1,y_1) and terminal point $B(x_2,y_2)$ then its position vector is $\vec{AB}=(x_2-x_1, y_2-y_1)= (x_2-x_1,)i + (y_2-y_1)j$
- Position vector whose initial point is origin(0,0) and terminal point is (x_2-x_1, y_2-y_1) This vector is $\vec{V}=(x_2-x_1,)i + (y_2-y_1)j$
- Any vector whose magnitude is unity is called unit vector. Generally for any vector V its unit vector is determined by dividing this vector by its magnitude. i.e

$$\vec{n_V}=\vec{V}/(/V/)$$

SCALARS AND VECTORS

- Vectors representing physical quantities can be classified as free, sliding, or fixed.
- A free vector is one whose action is not confined to or associated with a unique line in space. we may represent the displacement of a body by a free vector.
- A sliding vector has a unique line of action in space but not a unique point of application. For example, when an external force acts on a rigid body, the force can be applied at any point along its line of action without changing its effect on the body as a whole,* and thus it is a sliding vector.
- A fixed vector is one for which a unique point of application is specified. The action of a force on a deformable or nonrigid body must be specified by a fixed vector at the point of application of the force. In this instance the forces and deformations within the body depend on the point of application of the force, as well as on its magnitude and line of action.

Operation with Vectors

i) Resultant of vectors

- Vectors must obey the <u>parallelogram law of combination</u>. This law states that two vectors $\vec{V}_1$ and $\vec{V}_2$, treated as <u>free vectors</u>, Fig. 1/2a, may be replaced by their equivalent vector $\vec{V}$, which is the diagonal of the parallelogram formed by $\vec{V}_1$ and $\vec{V}_2$ as its two sides, as seen in Fig. 1/2b.This combination is called the vector sum, and is represented by the vector equation, $\vec{V} = \vec{V}_1 + \vec{V}_2$ The magnitude of

vector $\vec{V}$ calculated by the law of cosines as follows:

$/V/^2=/V_1/^2 + /V_2/^2 - 2*/V_1/*/V_2/*\cos(180-\theta)$ θ= angle between $\vec{V}_1$ & $\vec{V}_2$ when joined tail to tail as in Fig. 1/2b.

- The two vectors $\vec{V}_1$ and $\vec{V}_2$, again treated as free vectors, may also be

added head-to-tail by the <u>triangle law</u>, as shown in Fig. 1/2c, to obtain the identical vector sum V. Knowing θ, α and magnitudes of given vectors,$/V_1/$ and$/V_2/$ then the magnitude of Vector V, /V/ determined by using the law of sines as follows

$$\frac{(/V_1/)}{Sin\ \beta} = \frac{(/V_2/)}{Sin\ \alpha} = \frac{(/V/)}{Sin\ (180-\theta)}$$

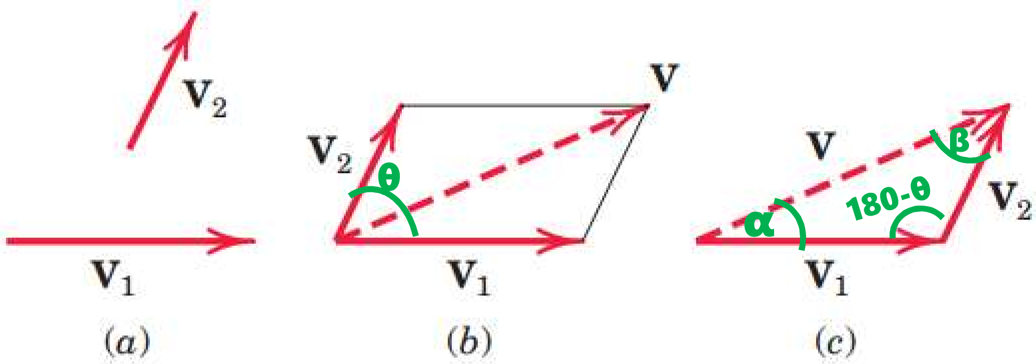


Figure 1/2

Operation with Vectors …

- The difference $\vec{V}_1$ - $\vec{V}_2$ between the two vectors is easily obtained by adding -$\vec{V}_2$ to $\vec{V}_1$ as shown in Fig. 1/3, where either the triangle or parallelogram procedure may be used. The difference $\vec{V}$\` between the two vectors is expressed by the vector equation $\vec{V}' = \vec{V}_1 - \vec{V}_2$

  where the minus sign denotes vector subtraction.

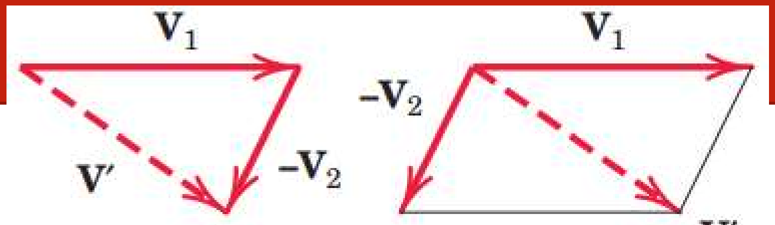


Figure 1/3

ii) Decomposition of a vector into its component for a given coordinate system

Any two or more vectors whose sum equals a certain vector V are said to be the components of that vector. Thus, the vectors $\vec{V}_1$ and $\vec{V}_2$ in Fig. 1/4a are the components of $\vec{V}$ in the directions 1 and 2, respectively. It is usually most convenient to deal with vector components which are mutually perpendicular; these are called rectangular components. The process of representing a vector by its component is called resolving a vector.

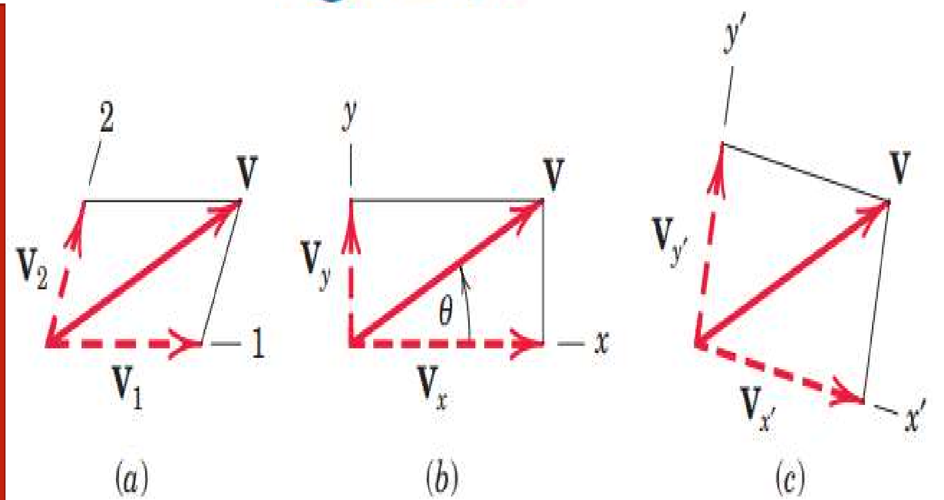
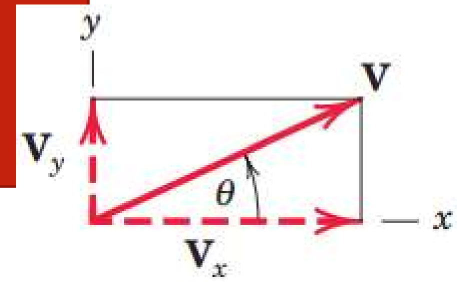


Figure 1/4

Operation with Vectors …

- **vectors V_x and V_y in Fig. 1/4b are the x- and y-components, respectively, of V. When expressed in rectangular components, the direction of the vector with respect to, say, the *x*-axis is clearly specified by the angle ϴ, where**



$$\theta = \tan^{-1} \frac{V_y}{V_x}$$

In many problems, particularly three-dimensional ones, it is convenient to express the rectangular components of **V**, Fig. 1/5, in terms of unit vectors **i**, **j**, and **k**, which are vectors in the *x*-, *y*-, and *z*-directions, respectively, with unit magnitudes. Because the vector **V** is the vector sum of the components in the *x*-, *y*-, and *z*-directions, we can express **V** as follows:

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$$

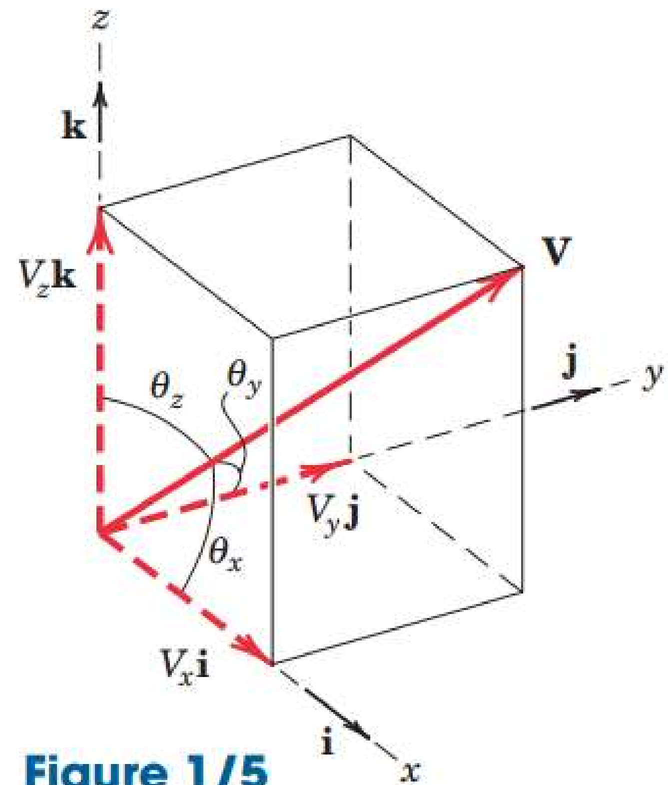


Figure 1/5

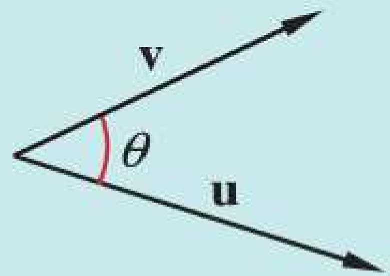
SCALARS AND VECTORS

Vector Multiplication: Dot & Cross

Dot Product

If **u** and **v** are vectors and θ is the angle between **u** and **v**, then the dot product of **u** and **v**, denoted by **u**·**v**, is defined by:

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta.$$



If U and V are non-zero vectors the angle between them can be calculated by:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$\mathbf{i}\cdot\mathbf{j} = 0$, $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1$

✓ If either **u** or **v** is **0**, then $\mathbf{u}\cdot\mathbf{v} = 0$.

$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$(dot product of vectors is commutative)

✓ If the vectors **u** and **v** are parallel, then $\mathbf{u}\cdot\mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$. In particular, for any vector **u**, we have $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$. Here, we write $\mathbf{u}^2$ to mean $|\mathbf{u}|^2$

✓ If the vectors **u** and **v** are perpendicular, then $\mathbf{u}\cdot\mathbf{v} = 0$ because $\cos\left(\frac{\pi}{2}\right) = 0$.

If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are vectors, then $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2$.

SCALARS AND VECTORS

Vector Multiplication: Dot & Cross

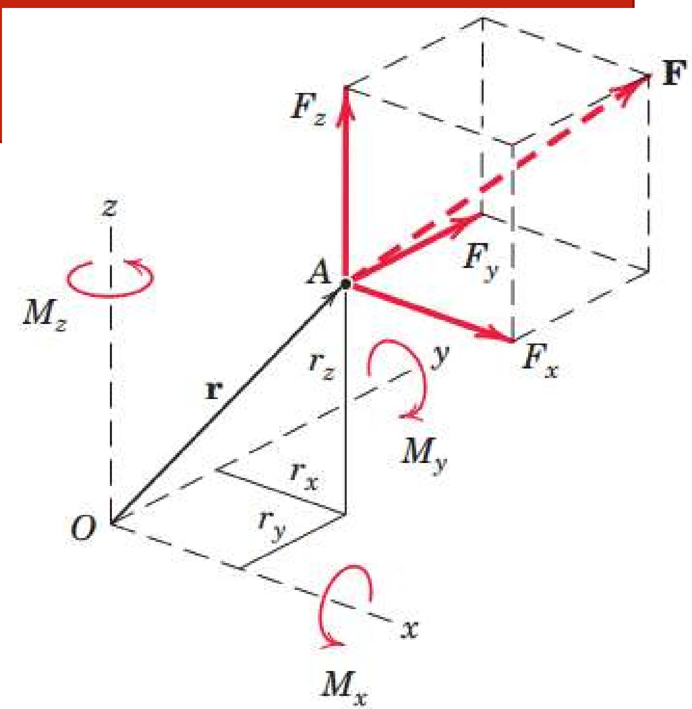
Cross Product

This vector multiplication is important in calculating moment of a force about an arbitrary point O. Let $\vec{r}$ is a position vector to the point of application of the force at A. That means $\vec{r}=\vec{OA}=r_x\,i+r_y\,j+r_z\,k$

If the force is written in the form of vector i.e $\vec{F}=F_x\,i+F_y\,j+F_z\,k$

Then the moment about O is the cross product of position vector and force vector, i.e $\vec{M_o}=\vec{r}\times\vec{F}$

- ✓ **Cross product is not commutative i.e $\vec{r}X\vec{F}=-\vec{F}X\vec{r}$**



$$M_o = rXF = \begin{vmatrix} i & j & k \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$M_o=(r_y*F_z-r_z*F_y)i-(r_x*F_z-r_z*F_x)j+(r_x*F_y-r_y*F_x)k$$

$$M_o=M_x\,i+M_y\,j+M_z\,k$$

Where M_x, M_y, & M_z, are the scalar component of the moment. The norm or magnitude of the moment can be calculated by as follows: $/Mo/=\sqrt{M^2{}_x+M^2{}_y+M^2{}_z}$

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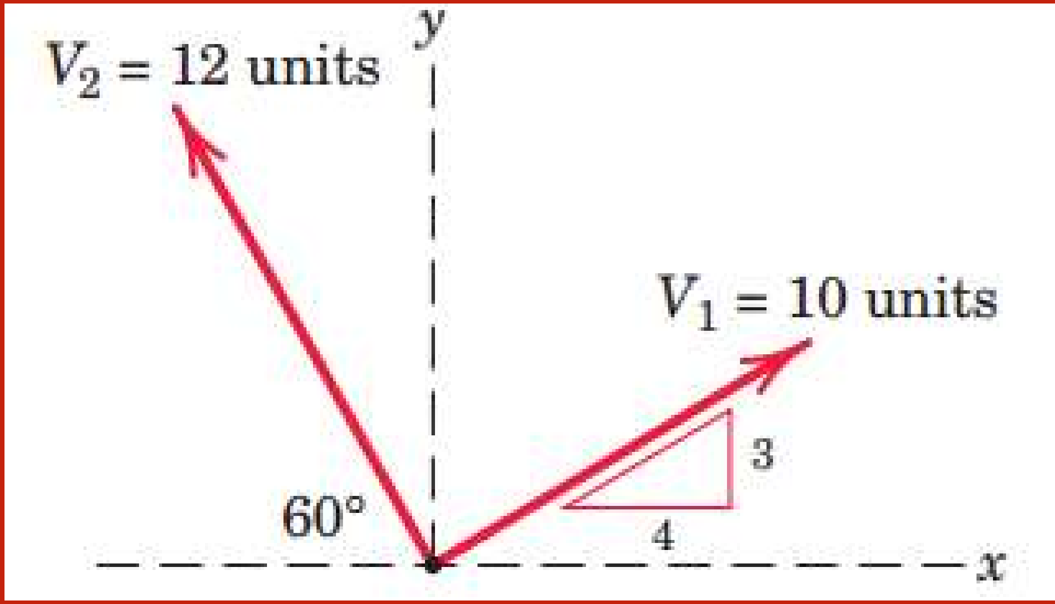
Exercise

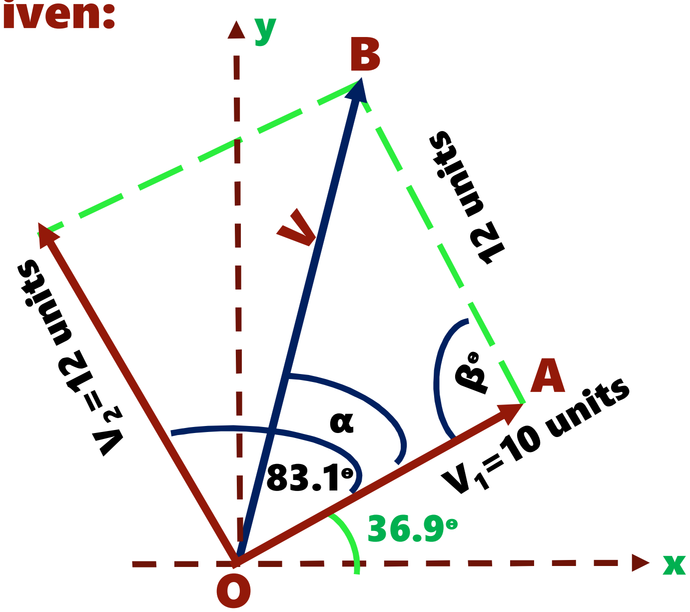
1. a) Determine the magnitude of the vector sum $V = V_1 + V_2$ and the angle θ_x which V makes with the positive x-axis.
 b) write V as a vector in terms of the unit vectors i and j
 c) Determine the unit vector of V.
 d) determine the magnitude of the vector difference $V` = V_2 - V_1$ and the angle $\theta_x$ which V` makes with the positive x-axis.



Solution

1. a) Required: $V=V_1+V_2$ and the angle θ_x which V makes with +ve x-axis.

Given:



Angle analysis

- The angle which V_1 makes with +ve x-axis=$\tan^{-1}(3/4)=36.9°$
- The angle b/n V1 and V2 =120 - 36.9=83.1° AND
- β=180-83.1=96.9°

1. Using Parallelogram law of vector addition see figure above,

The diagonal drawn is the vector sum, $V=V_1+V_2$.

- Take Triangle OAB. Magnitude of V, obtained by Using law of cosines

  $/V/^2= /V_1/^2+/V_2/^2 - 2/V_1/ * /V_2/\cos(96.9°)$

  $/V/^2= 10^2+12^2 - 2*10*12*\cos(96.9°)$, /V/=16.52 units…Answer

Then using law of sine $\frac{/V/}{Sin(96.9)}=\frac{/V2/}{Sin(\alpha)}$ w/c is $\frac{16.52}{Sin(96.9)}=\frac{12}{Sin(\alpha)}$

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Solving α, α=46.1 Thus θx=46.1+36.9=83° from positive x-axis …Answer

Solution

1. b) Required: Vector V in terms of i and j

$V = 16.52 * [\cos(83)\ i + \cos(90-83)\ j]$

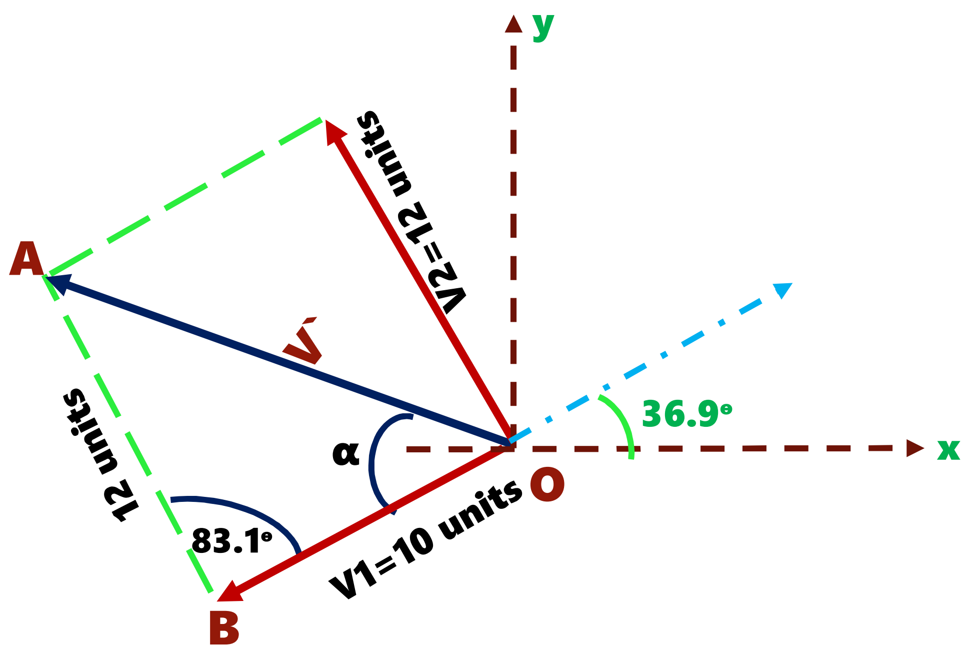
$= 2\ i + 16.34\ j$ (units)…Ans.

c) unit vector of V, $n_V = [2\ i + 16.34\ j]/16.52$

$= 0.12\ i + 0.99\ j$ ….Answer

Solution

d) Required: $V` = V_2 - V_1$ See the diagrammatic representation shown below.



- **Take Triangle OAB, Using law of cosines of triangles**

  i.e $/V`/^2 = /V_1/^2 + /V_2/^2 - 2/V_1/ * /V_2/\cos(83.1°)$

  $/V`/^2 = 10^2 + 12^2 - 2*10*12*\cos(83.1°)$ , $/V`/ = 14.67$ **units…Answer**

- **Then using law of sine** $\frac{/V`/}{Sin(83..1)} = \frac{/V2/}{Sin(\alpha)}$ **w/c** $\frac{14.67}{Sin(83.1)} = \frac{12}{Sin(\alpha)}$; $\alpha = 54.3°$

  Thus θx from negative x-axis=54.3 - 36.9=17.4°

  θx from Positive x-axis=180-17.4=162.6° ….Answer

2. FORCE SYSTEMS

Introduction

- ♠ Studying the effects of forces on structures is important in the study of mechanics and in other subjects such as stress analysis, design of structures and machines, and fluid flow.
- ♠ A force as an action of one body on another.
- ♠ In dynamics we will see that a force is defined as an action which tends to cause acceleration of a body.
- ♠ A force is a vector quantity, because its effect depends on the direction as well as on the magnitude of the action. Thus, forces may be combined according to the parallelogram law of vector addition.

Introduction…

- The action of the cable tension on the bracket in Fig. 2/1a is represented in the side view, Fig. 2/1b, by the force vector P of magnitude /P/. The effect of this action on the bracket depends on /P/, the angle Θ, and the location of the point of application A.

- Changing any one of these three specifications will alter the effect on the bracket, such as the force in one of the bolts which secure the bracket to the base, or the internal force and deformation in the material of the bracket at any point. Thus, the complete specification of the action of a force must include its magnitude, direction, and point of application, and therefore we must treat it as a fixed vector.

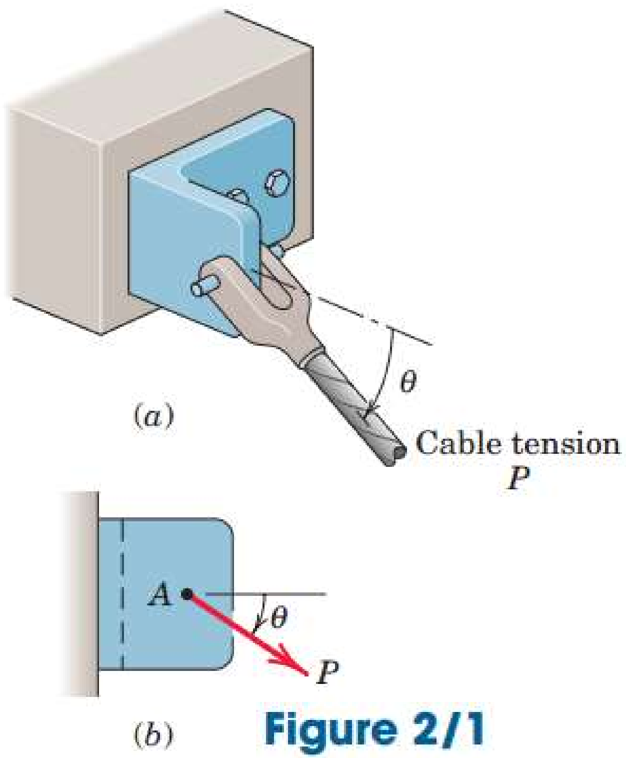


Figure 2/1

Introduction…

External and Internal Effects of forces

- **External effects**

Due to external applied force on an object, there will be a reaction force to be developed from the contacting surface.

Thus force external to the body are:

- ♠ Applied forces
- ♠ Reactive forces (support reaction force).

- **Internal effects**
- **The effects of a force internal to a given body are:**

➢ internal forces (Axial and shear stress, moments) and

➢ deformations or deflections (strain)

Introduction…

Principle of Transmissibility

- For example, the force P acting on the rigid plate in Fig. 2/2 may be applied at A or at B or at any other point on its line of action, and the net external effects of P on the bracket will not change. The external effects are the force exerted on the plate by the bearing support at O and the force exerted on the plate by the roller support at C.

- This conclusion is summarized by the principle of transmissibility, which states that a force may be applied at any point on its given line of action without altering the resultant effects of the force external to the rigid body on which it acts.

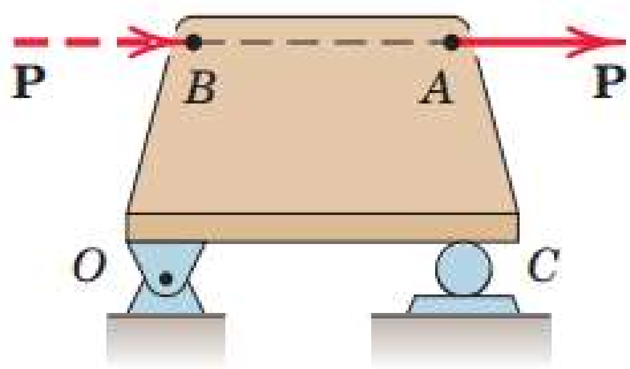


Figure 2/2

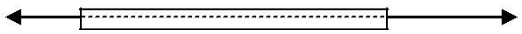
Introduction…

Force Classification

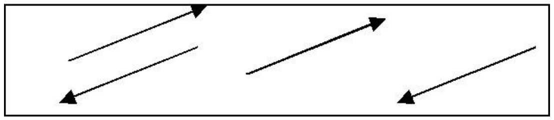
- Forces are classified as either contact or body forces.
  - A contact force is produced by direct physical contact; an example is the force exerted on a body by a supporting surface.
  - A body force is generated by virtue of the position of a body within a force field such as a gravitational, electric, or magnetic field. E.g: Weight of a given object.
- Forces may be further classified as either
  - concentrated forces or point load:
  - Distributed:
    - Linearly distributed or line load
    - Areally distributed

Introduction

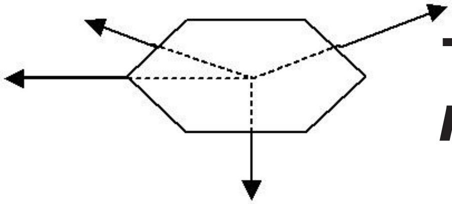
- A system of forces can be grouped into different categories depending on their arrangement in space.
  - Coplanar Forces:-are forces which act on the same plane.
- Depending on their arrangement on the plane too, coplanar forces can further be divided as:
  - Coplanar collinear forces:-are coplanar forces acting on the same line-collinear.



  - Coplanar parallel forces:-Are forces which are on the same plane and parallel

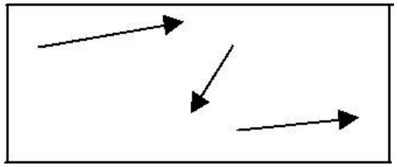


  - Coplanar concurrent forces:-Are forces on the same plane whose lines of action intersect at a point.



Two or more forces are said to be *concurrent at a point* if their lines of action intersect at that point.

  - General coplanar forces:



SECTION A TWO-DIMENSIONAL FORCE SYSTEMS

Rectangular Components

- **The most common two-dimensional resolution of a force vector is into rectangular components. It follows from the parallelogram rule that the vector F of Fig. 2/5 may be written as** $\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y$

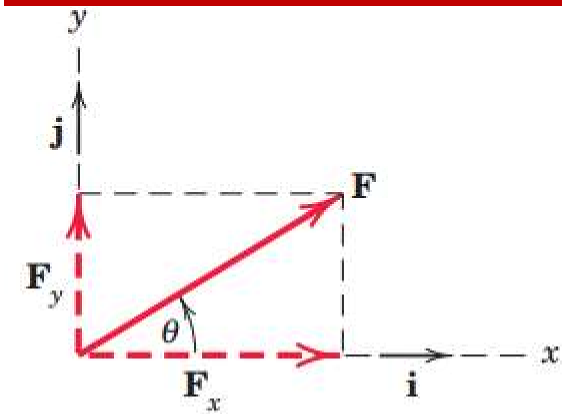


Figure 2/5

where $\mathbf{F}_x$ and $\mathbf{F}_y$ are *vector components* of F in the *x*- and *y*-directions. Each of the two vector components may be written as a scalar times the appropriate unit vector. In terms of the unit vectors **i** and **j** of Fig. 2/5, $\mathbf{F}_x = F_x\mathbf{i}$ and $\mathbf{F}_y = F_y\mathbf{j}$, and thus we may write $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$

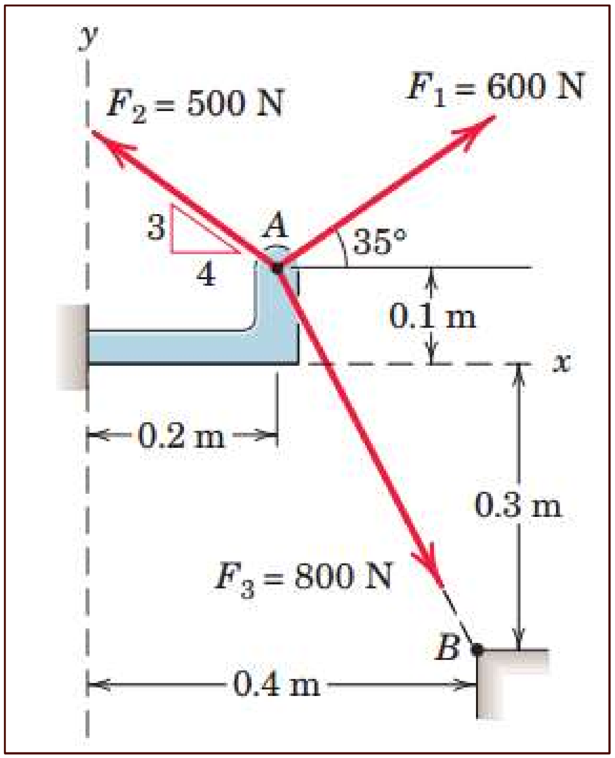
where the scalars F_x and F_y are the *x* and *y scalar components* of the vector **F**. For the force vector of Fig. 2/5, the *x* and *y* scalar components are both positive and are related to the magnitude and direction of **F** by

$$F_x = F\cos\theta \qquad F = \sqrt{F_x^{\,2} + F_y^{\,2}}$$

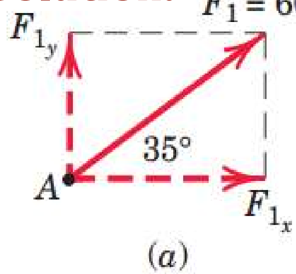
$$F_y = F\sin\theta \qquad \theta = \tan^{-1}\frac{F_y}{F_x}$$

Example

The forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, all of which act on point *A* of the bracket, are specified in three different ways. Determine the *x* and *y* scalar components of each of the three forces **and Compute the resultant force, R.**

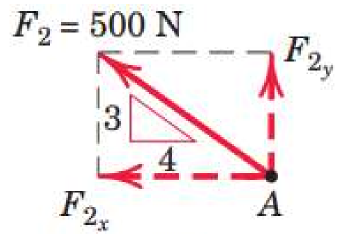


Solution:



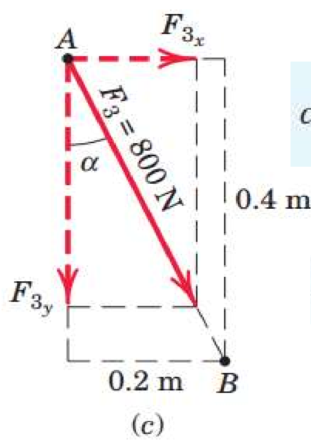
$F_{1_x} = 600 \cos 35° = 491$ N

$F_{1_y} = 600 \sin 35° = 344$ N



$F_{2_x} = -500(\frac{4}{5}) = -400$ N

$F_{2_y} = 500(\frac{3}{5}) = 300$ N



$$\alpha = \tan^{-1}\left[\frac{0.2}{0.4}\right] = 26.6°$$

$F_{3_x} = F_3 \sin\alpha = 800 \sin 26.6° = 358$ N

$F_{3_y} = -F_3 \cos\alpha = -800 \cos 26.6° = -716$ N

$$\mathbf{F}_3 = F_3\mathbf{n}_{AB} = 800\left[\frac{0.2\mathbf{i} - 0.4\mathbf{j}}{\sqrt{(0.2)^2 + (-0.4)^2}}\right]$$
$$= 800\,[0.447\mathbf{i} - 0.894\mathbf{j}]$$
$$= 358\mathbf{i} - 716\mathbf{j}\ \text{N}$$

Finally:-

$$\sum Fx = 491 - 400 + 358 = 449N$$

$$\sum Fy = 344 + 300 - 716 = -72N$$

R=449 i-72 j (N) ….Answer

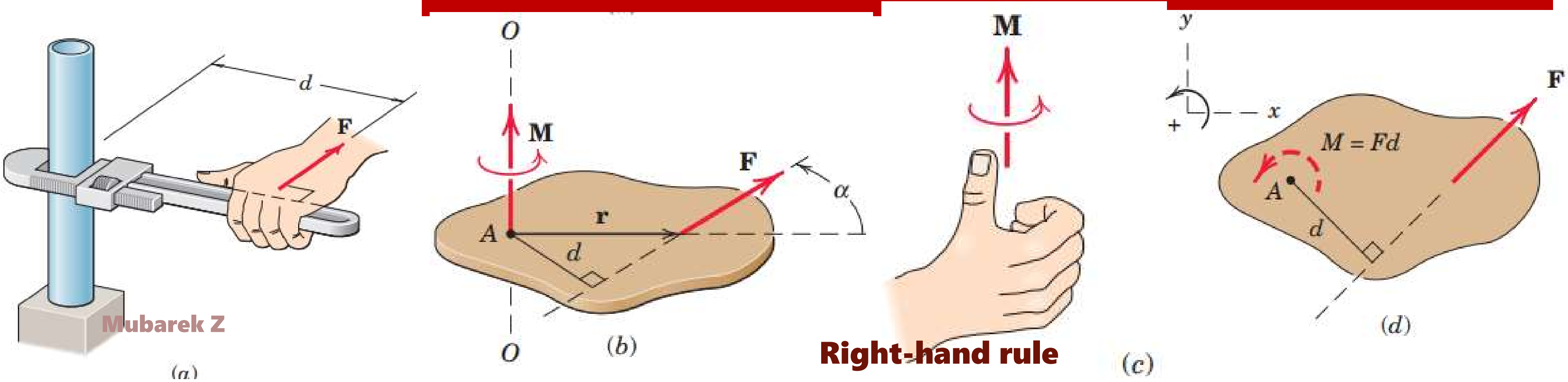
2D System …

<u>Moment</u>

- In addition to the tendency to move a body in the direction of its application, a force can also tend to rotate a body about an axis. The axis may be any line which neither intersects nor is parallel to the line of action of the force. This rotational tendency is known as the moment M of the force. Moment is also referred to as torque.

Moment about a Point

- When dealing with forces which all act in a given plane, we customarily speak of the moment about a point. By this we mean the moment with respect to an axis normal to the plane and passing through the point. Thus, the moment of force F about point A in Figure (d) has the magnitude M = Fd and is counterclockwise.
- Moment directions may be accounted for by using a stated sign convention, such as a plus sign (+) for counterclockwise moments and a minus sign (-) for clockwise moments, or vice versa.



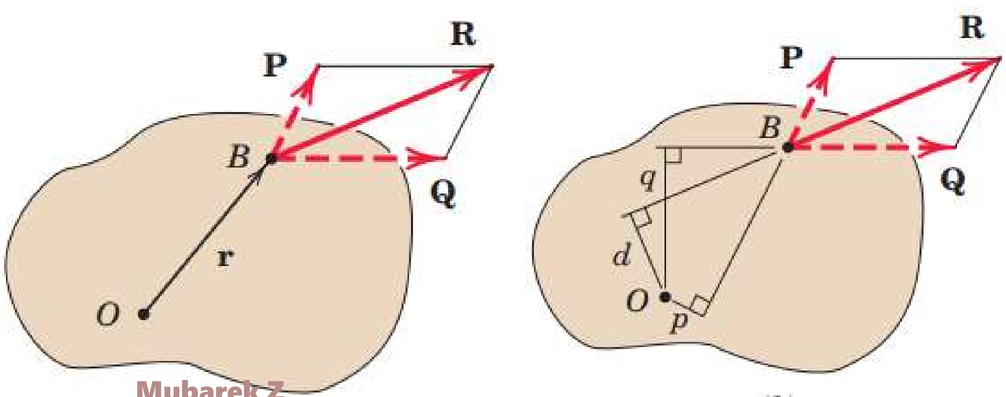
2D System …

Moment Calculation by:

1) **Moment arm rule: Moment is the scalar product of force magnitude with a distance which is perpendicular to the line of action of the force, i.e M=Fd**
2) **Vector Approach: Use of the Cross-Product M=rXF where r is Position vector locating the point of application of the force vector F. Where F is a force written in the form of vector. We establish the direction and sense of M by applying the right-hand rule to the sequence rXF. If the fingers of the right hand are curled in the direction of rotation from the positive sense of r to the positive sense of F, then the thumb points in the positive sense of M.**

Varignon's Theorem

One of the most useful principles of mechanics is Varignon's theorem, which states that the moment of a force about any point is equal to the sum of the moments of the components of the force about the same point.



(a) (b)

By Moment arm rule:

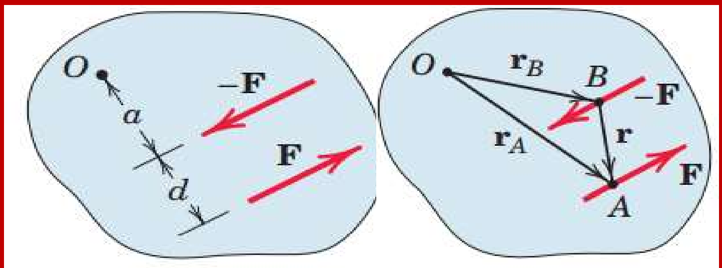
$$M_O = Rd = -pP + qQ$$

By Vector approach: Using the distributive law for cross products, we have $\mathbf{M}_O = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times \mathbf{P} + \mathbf{r} \times \mathbf{Q}$

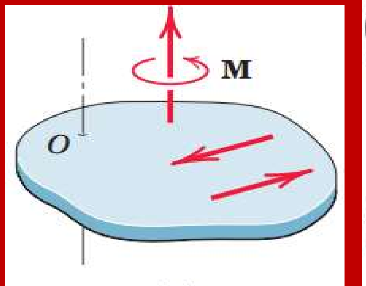
2D System …

Couple

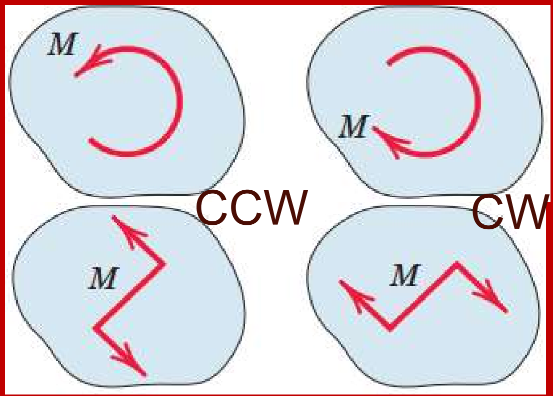
- The moment produced by two equal, opposite, and noncollinear forces is called a couple. Couples have certain unique properties and have important applications in mechanics.



(a) (b)



(c)



- Here again, the moment expression contains no reference to the moment center O and, therefore, is the same for all moment centers. Thus, we may represent M by a free vector, as shown in Fig. 2/10c, where the direction of M is normal to the plane of the couple and the sense of M is established by the right-hand rule.

2D System …

Equivalent Couples

- Changing the values of F and d does not change a given couple as long as the product Fd remains the same. Likewise, a couple is not affected if the forces act in a different but parallel plane.
- Figure 2/11 shows four different configurations of the same couple M. In each of the four cases, the couples are equivalent and are described by the same free vector which represents the identical tendencies to rotate the bodies.

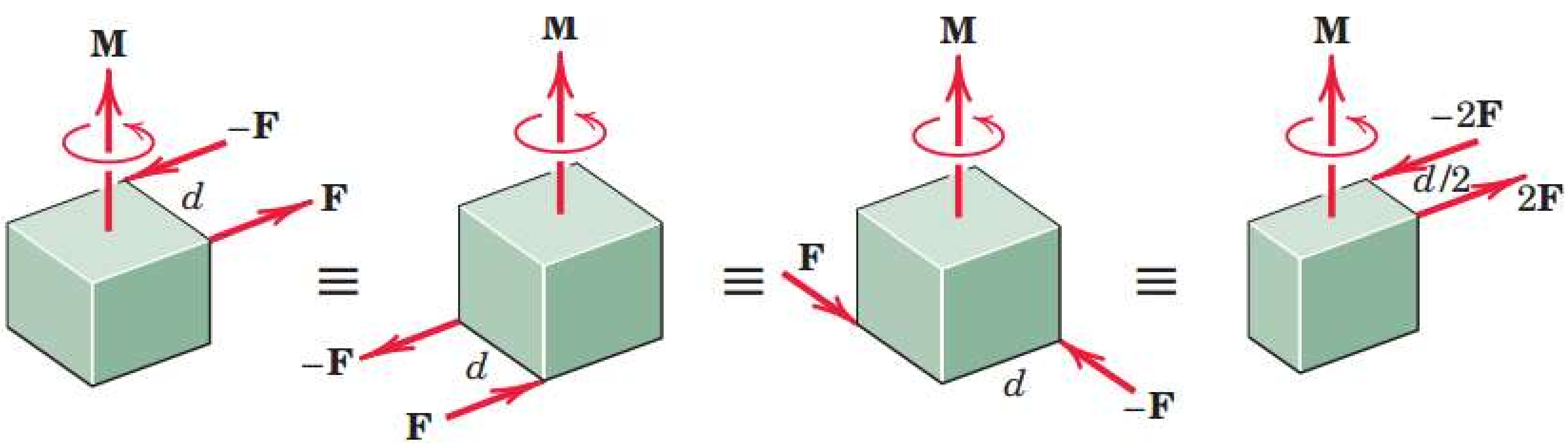


Figure 2/11

2D System …

Force-couple System

- The effect of a force acting on a body is the tendency to push or pull the body in the direction of the force, and to rotate the body about any fixed axis which does not intersect the line of the force. We can represent this dual effect more easily by replacing the given force by an equal parallel force and a couple to compensate for the change in the moment of the force.
- The replacement of a force by a force and a couple is illustrated in Fig. 2/12, where the given force F acting at point A is replaced by an equal force F at some point B and the counterclockwise couple M =F*d. The transfer is seen in the middle figure, where the equal and opposite forces F and -F are added at point B without introducing any net external effects on the body. The combination of the force and couple in the right-hand part of Fig. 2/12 is referred to as a force–couple system.

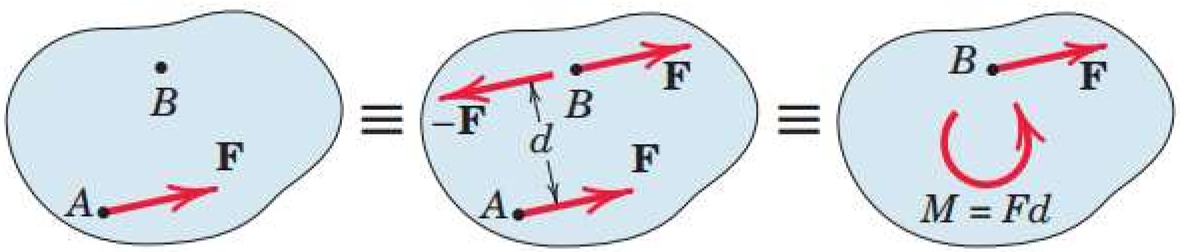


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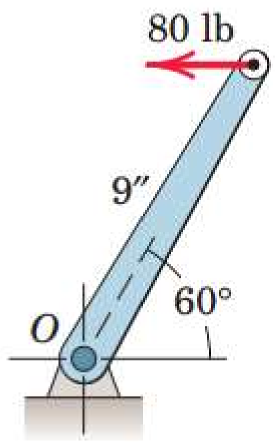
Figure 2/12

2D System …

Force-couple System…

- By reversing this process, we can combine a given couple and a force which lies in the plane of the couple (normal to the couple vector) to produce a single, equivalent force. Replacement of a force by an equivalent force–couple system, and the reverse procedure, have many applications in mechanics and should be mastered

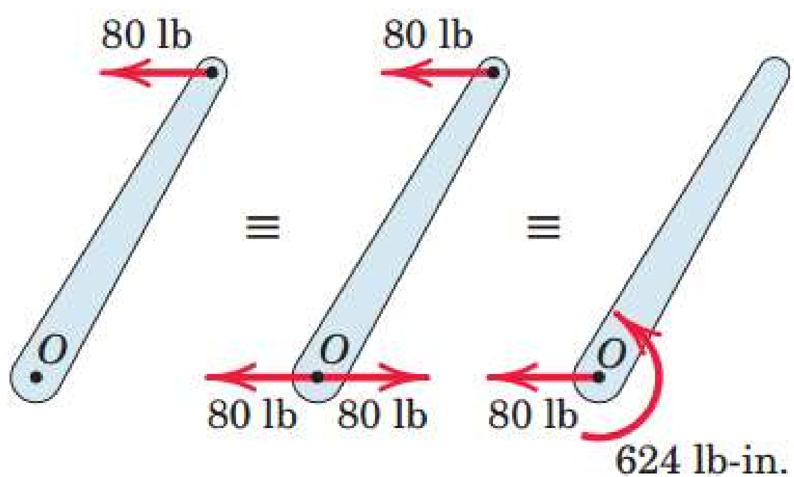
Example: Replace the horizontal 80-lb force acting on the lever by an equivalent system consisting of a force at O and a couple.



Solution:

We apply two equal and opposite 80-lb forces at O and identify the counterclockwise couple

[M =F*d];
M = 80(9 sin 60) = 624 lb-in.



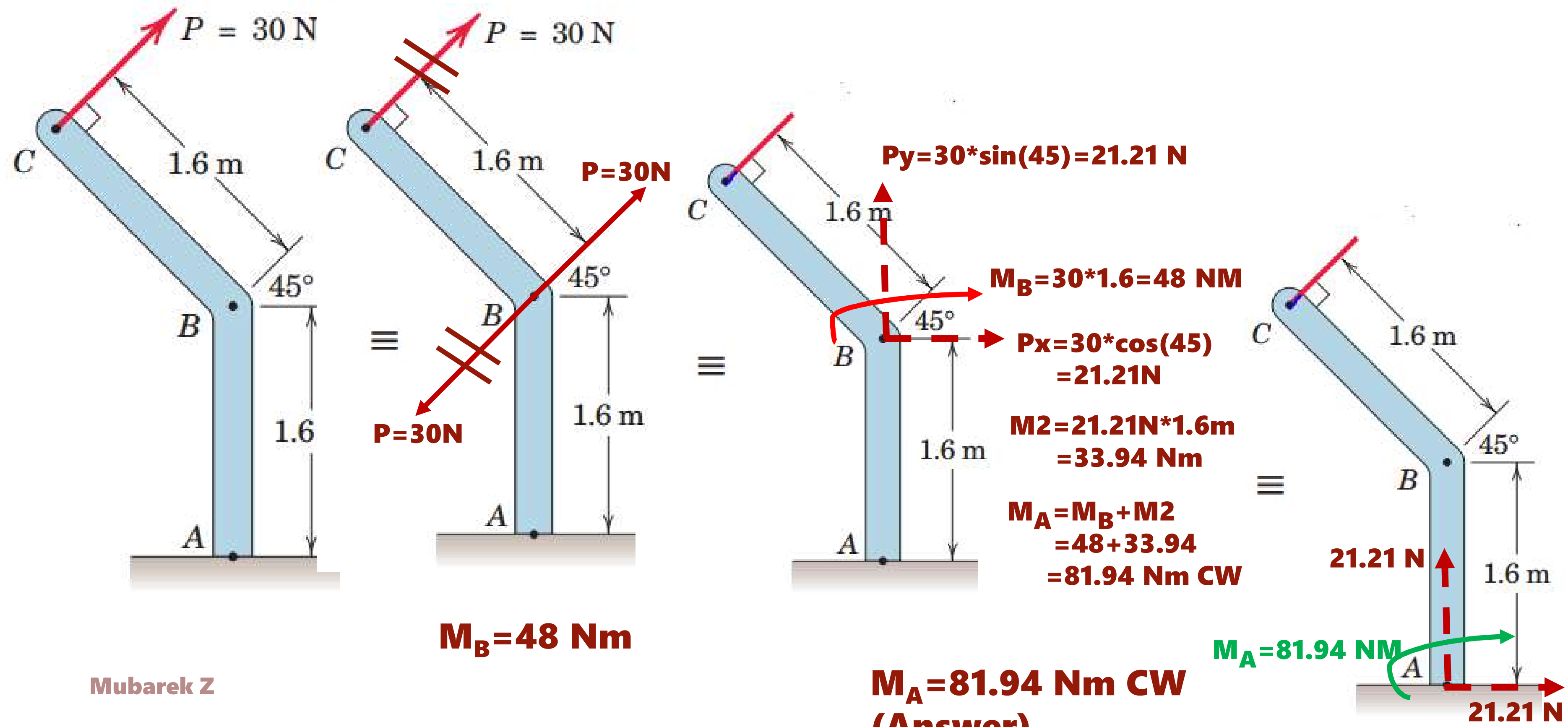
Answer

2D Examples…

Force-coup System…

- **The 30-N force P is applied perpendicular to the portion BC of the bent bar. Determine the moment of P about point B and about point A.**

Solution:



M_B=48 Nm

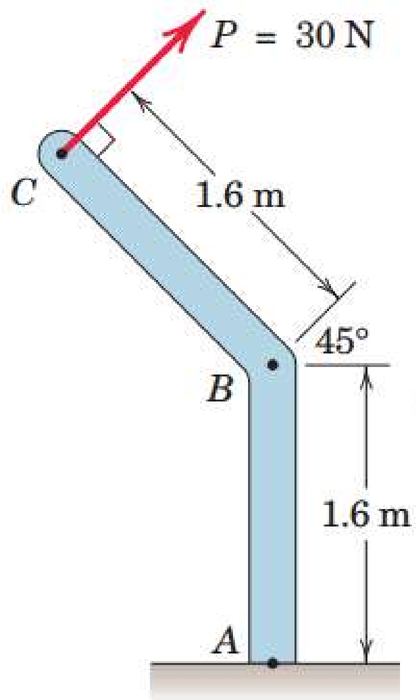
M_A=81.94 Nm CW (Answer)

2D Examples…

Force-coup System…

- **The 30-N force P is applied perpendicular to the portion BC of the bent bar. Force-couple system at point A.**

Given:

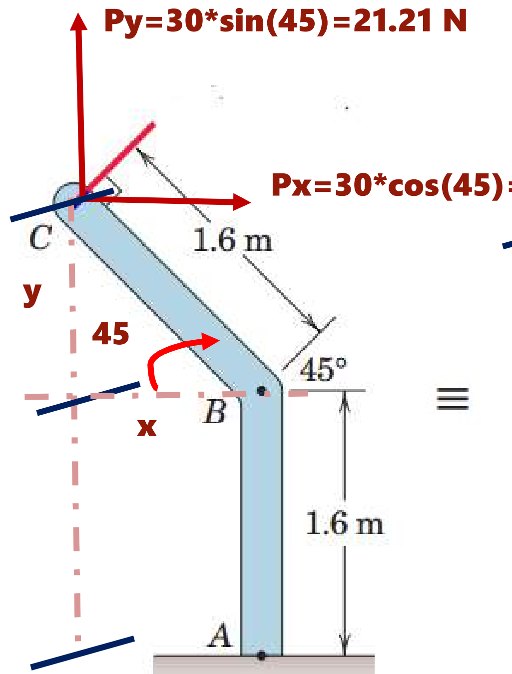


≡

Solution:

x=1.6*cos45=1.131 m

y=1.6*sin45=1.131 m



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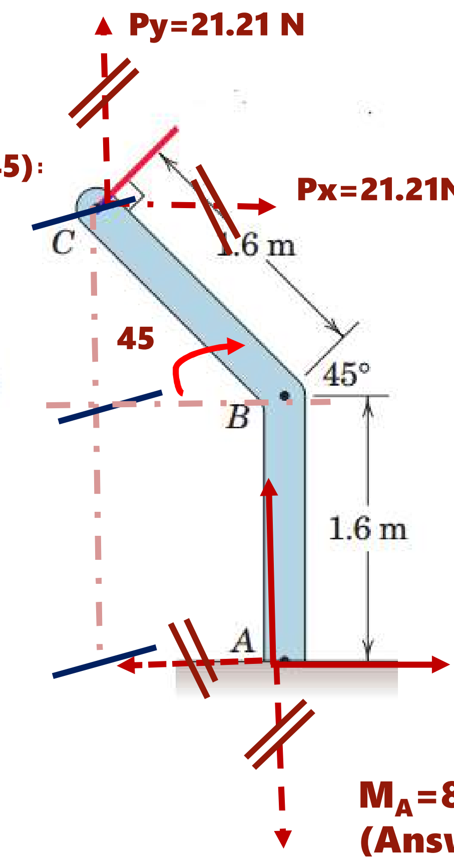
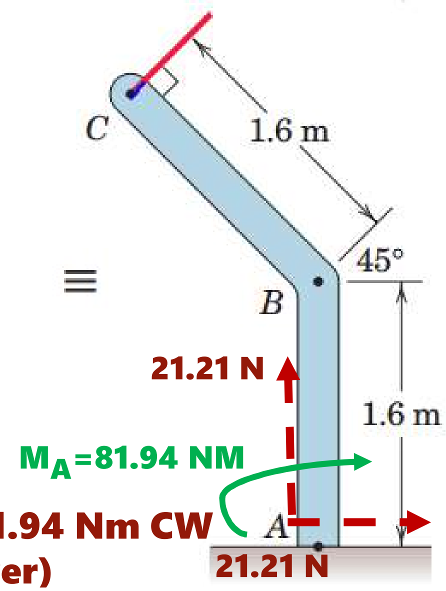


M_{Py}=21.21*1.131=23.99 Nm

M_{Px}=21.21*(1.131+1.6)
=57.93 Nm

M_A=M_{Py}+M_{Px} =81.94 Nm

≡

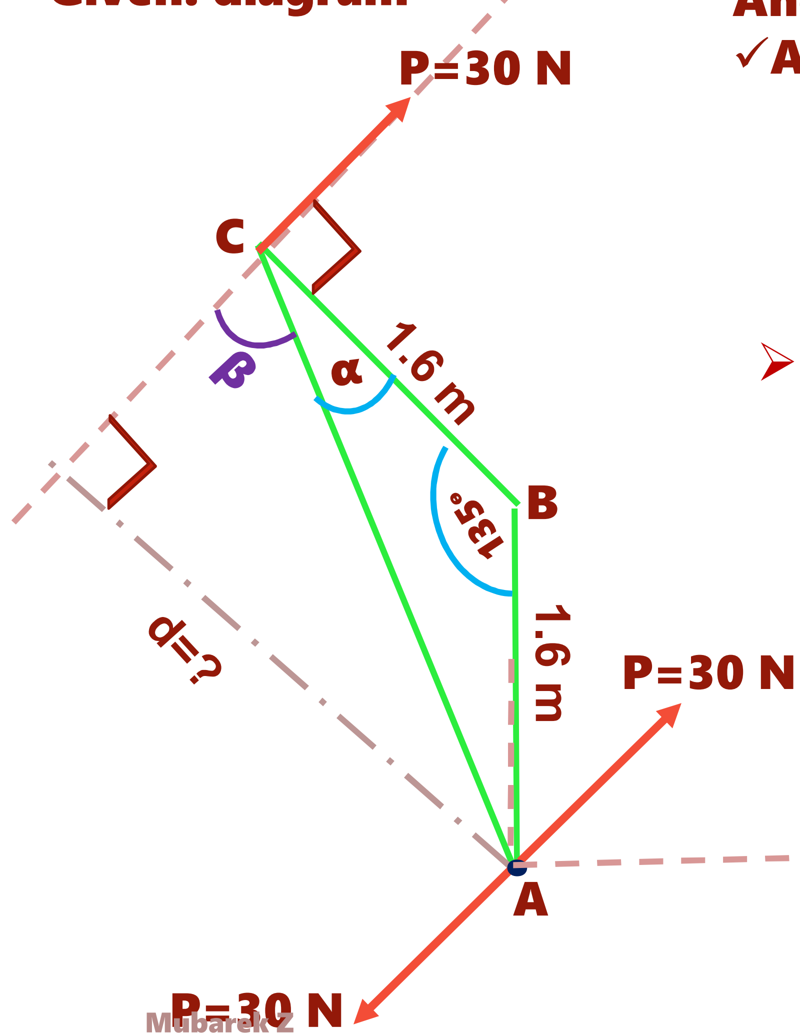


M_A=81.94 Nm CW
(Answer)

2D Examples…

Solution: Alternative approach by calculating moment arm

Given: diagram



Angle analysis:

✓ **Angle <ABC=180-45=135° and α=?**

Dimension Analysis:

✓ **Using Cosine laws for the triangle ABC:**

$/AC/^2=1.6^2 + 1.6^2 - 2*1.6*1.6*\cos(135°)$,

/AC/=2.956m and then Using sine law:

➢ $\frac{/AC/}{Sin(135)}=\frac{/AB/}{Sin(\alpha)}$ thus $\frac{2.956}{Sin(135)}=\frac{1.6}{Sin(\alpha)}$ **; α=22.5°**

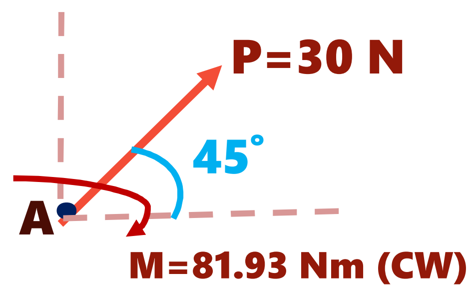
Therefore β=90-α; β=90-22.5; β=67.5°

✓ **d=/AC/*sin β=2.956*sin(67.5°)=2.73m.**

✓ **Then by applying two equal but opposite forces of magnitude=30N at point A,**

✓ **As a result**

- **Couple, M=30*2.73=81.93 Nm (CW) and**
- **Force, P=30 N at 45° from horizontal line**



(Answer)

2D System …

Resultant

- Most problems in mechanics deal with a system of forces, and it is usually necessary to reduce the system to its simplest form to describe its action. The resultant of a system of forces is the simplest force combination which can replace the original forces without altering the external effect on the rigid body to which the forces are applied.

- Equilibrium of a body is the condition in which the resultant of all forces acting on the body is zero. This condition is studied in statics. When the resultant of all forces on a body is not zero, the acceleration of the body is obtained by equating the force resultant to the product of the mass and acceleration of the body. This condition is studied in dynamics. Thus, the determination of resultants is basic to both statics and dynamics.

2D System …

Resultant…

- **The most common type of force system occurs when the forces all act in a single plane, say, the x-y plane, as illustrated by the system of three forces F1, F2, and F3 in Fig. 2/13a. We obtain the magnitude and direction of the resultant force R by forming the force polygon shown in part b of the figure, where the forces are added head-to-tail in any sequence. Thus, for any system of coplanar forces we may write**

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \Sigma\mathbf{F}$$

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$\theta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{\Sigma F_y}{\Sigma F_x}$$

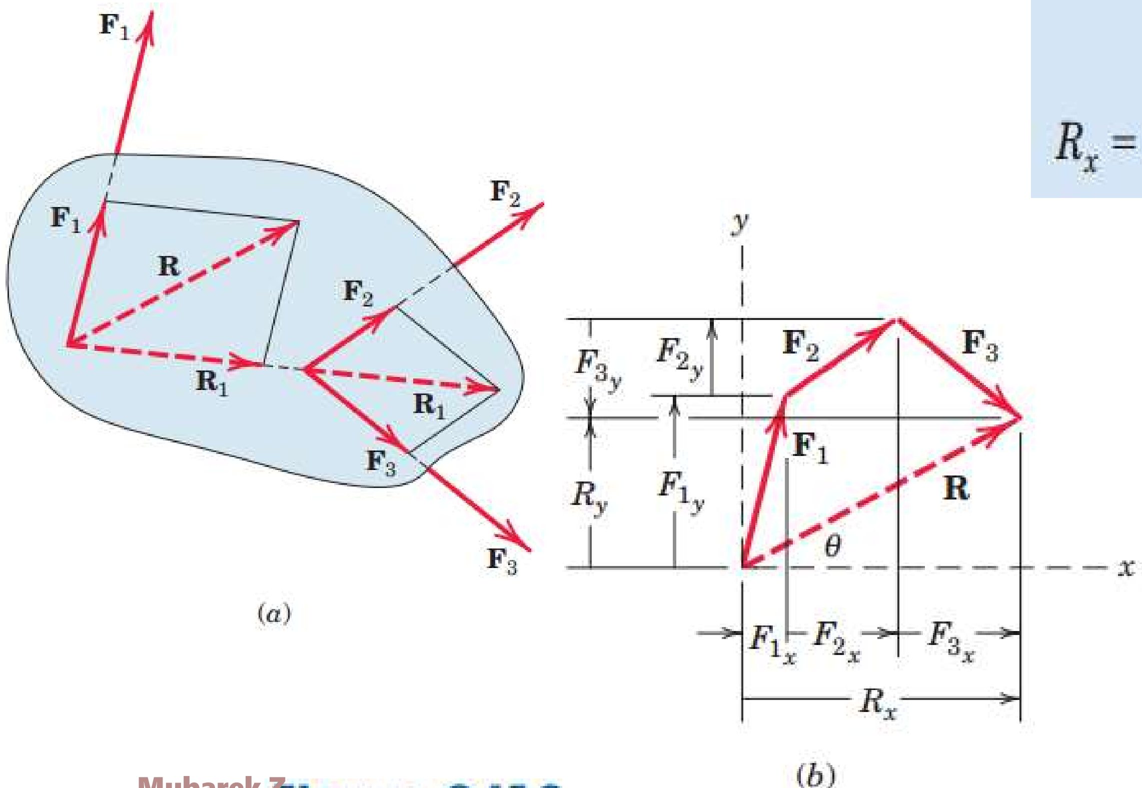


Figure 2/13

Graphically, the correct line of action of R may be obtained by preserving the correct lines of action of the forces and adding them by the parallelogram law. The principle of transmissibility usually used in this process.

2D System …

Resultant…

Algebraic Method: ***the magnitude and direction of the resultant force (R) for the given force system by a vector summation of forces . The steps are:***

1. **Choose a convenient reference point and move all forces to that point. This process is depicted for a three-force system in Figs. 2/14a and b, where M1, M2, and M3 are the couples resulting from the transfer of forces F1, F2, and F3 from their respective original lines of action to lines of action through point O.**
2. **Add all forces at O to form the resultant force R, & add all couples to form the resultant couple MO. We now have the single force–couple system, as shown in Fig. c.**

3. In Fig. 2/14d, find the line of action of R by requiring R to have a moment of M_O about point O. Note that the force systems of Figs. 2/14a and 2/14d are equivalent, and that Σ(Fd) in Fig. 2/14a is equal to Rd in Fig. 2/14d.

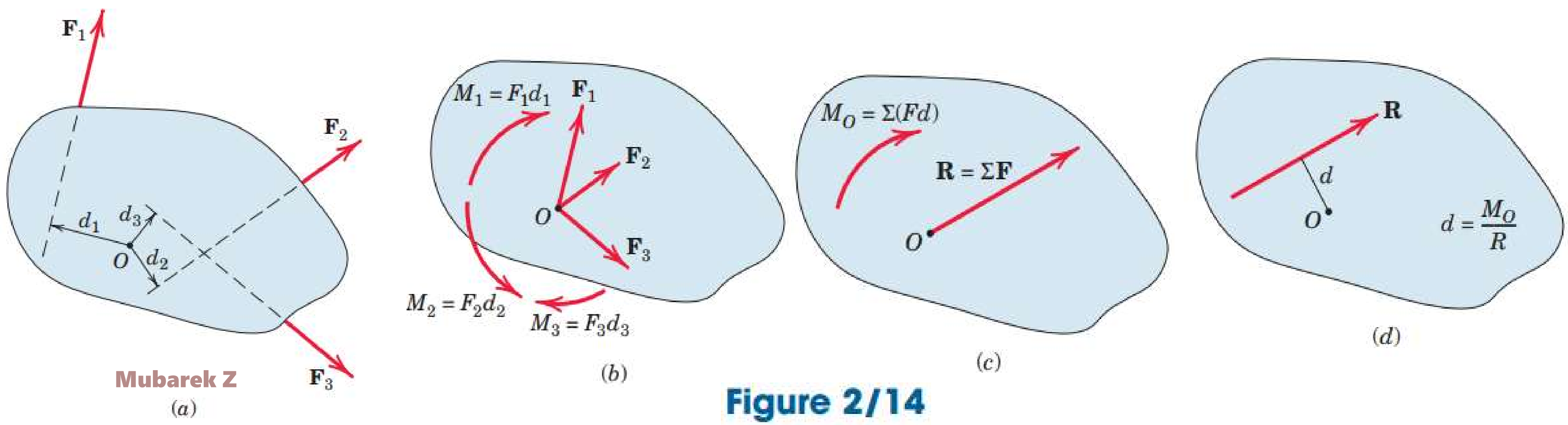
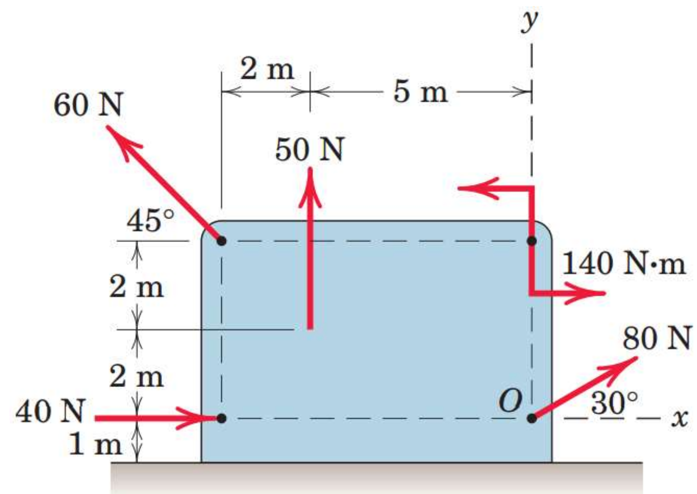


Figure 2/14

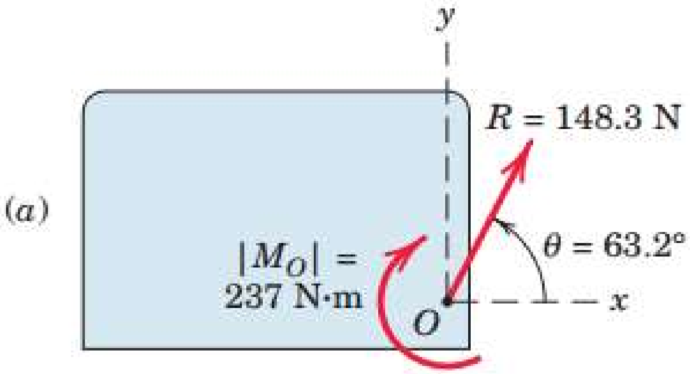
2D System …

Example

- **We Determine the resultant of the four forces and one couple which act on the plate shown.**



Solution: Point *O* is selected as a convenient reference point for the force–couple system which is to represent the given system.



$[R_x = \Sigma F_x]$ $\quad R_x = 40 + 80\cos 30° - 60\cos 45° = 66.9\text{ N}$

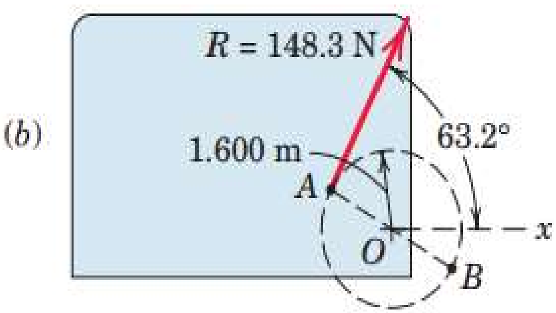
$[R_y = \Sigma F_y]$ $\quad R_y = 50 + 80\sin 30° + 60\cos 45° = 132.4\text{ N}$

$[R = \sqrt{R_x^2 + R_y^2}]$ $\quad R = \sqrt{(66.9)^2 + (132.4)^2} = 148.3\text{ N}$

$\left[\theta = \tan^{-1}\dfrac{R_y}{R_x}\right]$ $\quad \theta = \tan^{-1}\dfrac{132.4}{66.9} = 63.2°$

$[M_O = \Sigma(Fd)]$ $\quad M_O = 140 - 50(5) + 60\cos 45°(4) - 60\sin 45°(7)$

$= -237\text{ N}\cdot\text{m}$



Also determine the final line of action of R such that R alone represents the original system. $[Rd = |M_O|]$ $\quad 148.3d = 237$ $\quad d = 1.600\text{ m}$

Generally, the desired line of action by principle of moments:-

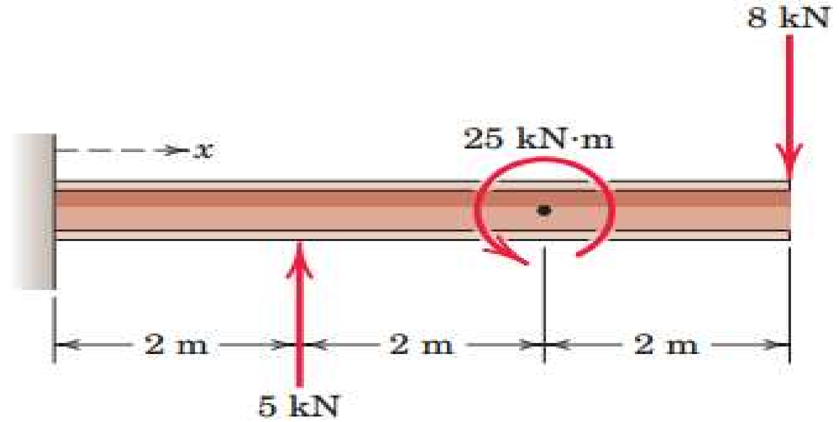
$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j}$

$(x\mathbf{i} + y\mathbf{j}) \times (66.9\mathbf{i} + 132.4\mathbf{j}) = -237\mathbf{k}$ $\quad (132.4x - 66.9y)\mathbf{k} = -237\mathbf{k}$ $\quad 132.4x - 66.9y = -237$

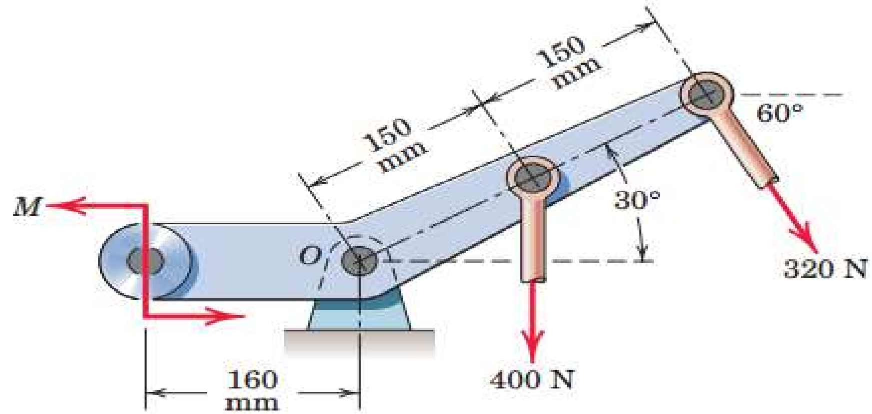
2D System …

Exercise

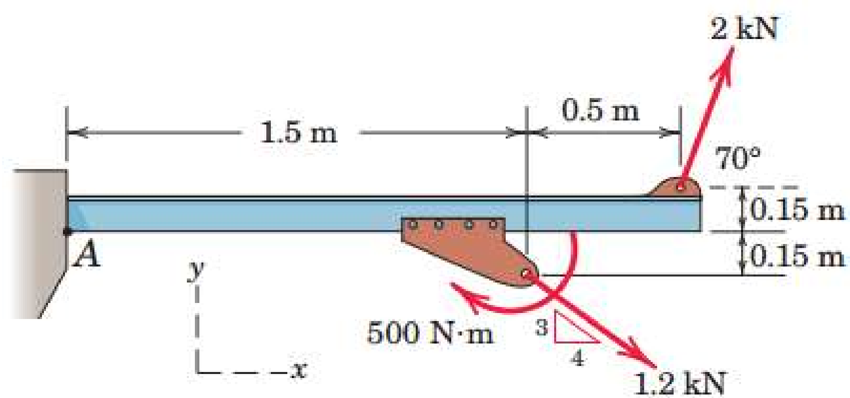
1) **Determine and locate the resultant R of the two forces and one couple acting on the I-beam.**



2) **If the resultant of the two forces and couple M passes through point O, determine M.**



3) **Replace the two forces and couple by an equivalent couple M and resultant force R at A.**



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3D Systems

Rectangular Components

- Many problems in mechanics require analysis in three dimensions, and for such problems it is often necessary to resolve a force into its three mutually perpendicular components.
- The force F acting at point O in Fig. 2/16 has the rectangular components Fx, Fy, Fz, where

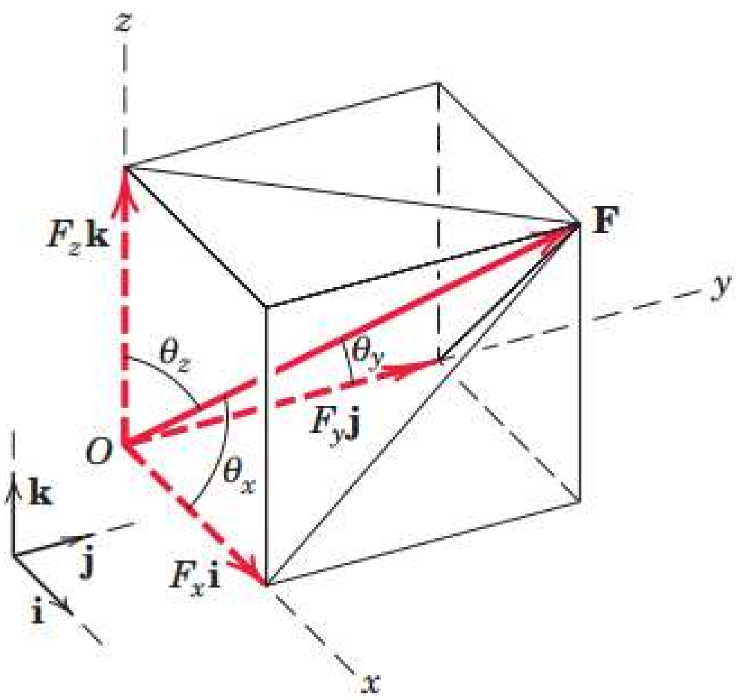


Figure 2/16

$$F_x = F\cos\theta_x \qquad F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

$$F_y = F\cos\theta_y \qquad \mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

$$F_z = F\cos\theta_z \qquad \mathbf{F} = F(\mathbf{i}\cos\theta_x + \mathbf{j}\cos\theta_y + \mathbf{k}\cos\theta_z)$$

The unit vectors **i**, **j**, and **k** are in the *x*-, *y*-, and *z*-directions, respectively. Using the direction cosines of **F**, which are $l = \cos\theta_x$, $m = \cos\theta_y$, and $n = \cos\theta_z$, where $l^2+m^2+n^2=1$, we may write the force as $\mathbf{F} = F(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$

Force, **F** equals the force magnitude *F* times a unit vector $\mathbf{n}_F$ which characterizes the direction of **F**, or

$$\mathbf{F} = F\mathbf{n}_F$$

3D System …

Force-couple System

- **The In solving three-dimensional problems, one must usually find the x, y, and z scalar components of a force. In most cases, the direction of a force is described:**
  - **(a) by two points on the line of action of the force or**
  - **(b) by two angles which orient the line of action.**

(a) Specification by two points on the line of action of the force.

If the coordinates of points *A* and *B* of Fig. 2/17 are known, the force **F** may be written as

$$\mathbf{F} = F\mathbf{n}_F = F\frac{\overrightarrow{AB}}{AB} = F\frac{(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

Thus the *x*, *y*, and *z* scalar components of **F** are the scalar coefficients of the unit vectors **i**, **j**, and **k**, respectively.

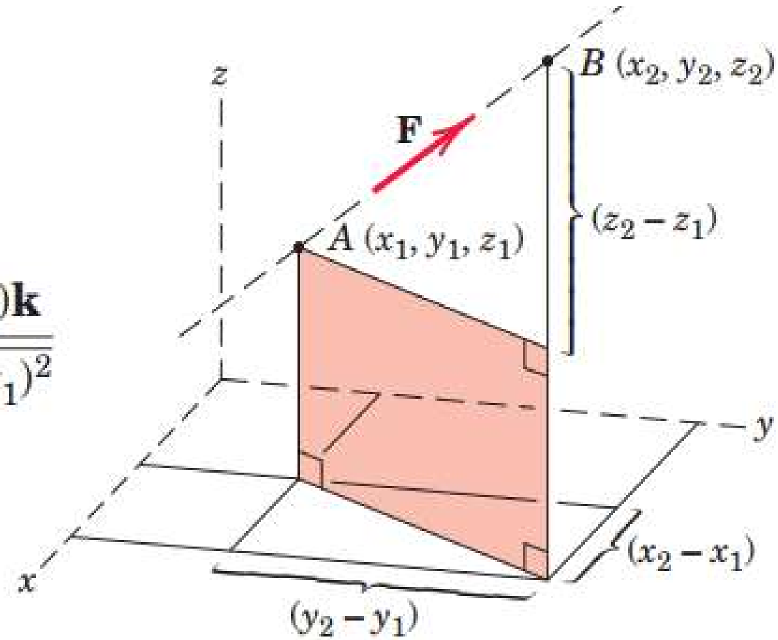


Figure 2/17

3D System …

(b) Specification by two angles which orient the line of action of the force.

Consider the geometry of Fig. 2/18. We assume that the angles ϴ and Ø are known. First resolve **F** into horizontal and vertical components.

$$F_{xy} = F \cos \phi$$

$$F_z = F \sin \phi$$

Then resolve the horizontal component F_{xy} into *x*- and *y*-components

$$F_x = F_{xy} \cos \theta = F \cos \phi \cos \theta$$

$$F_y = F_{xy} \sin \theta = F \cos \phi \sin \theta$$

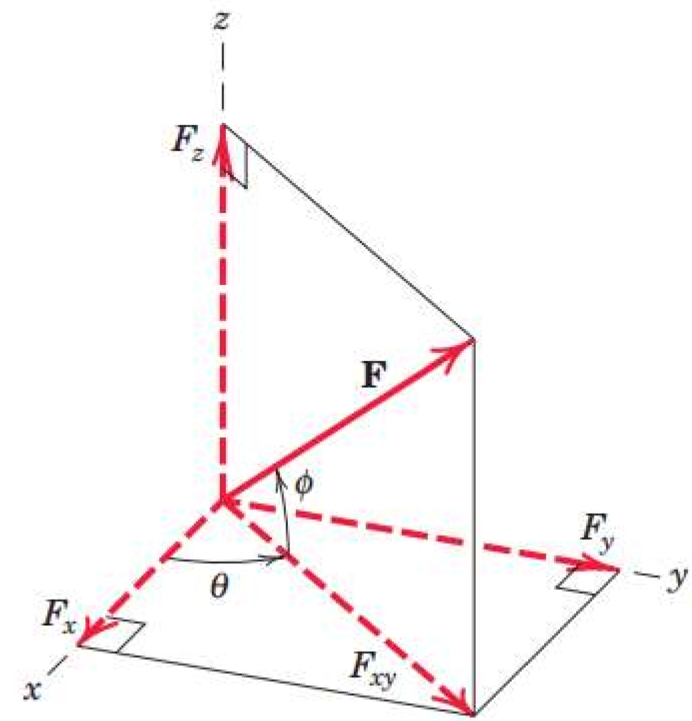
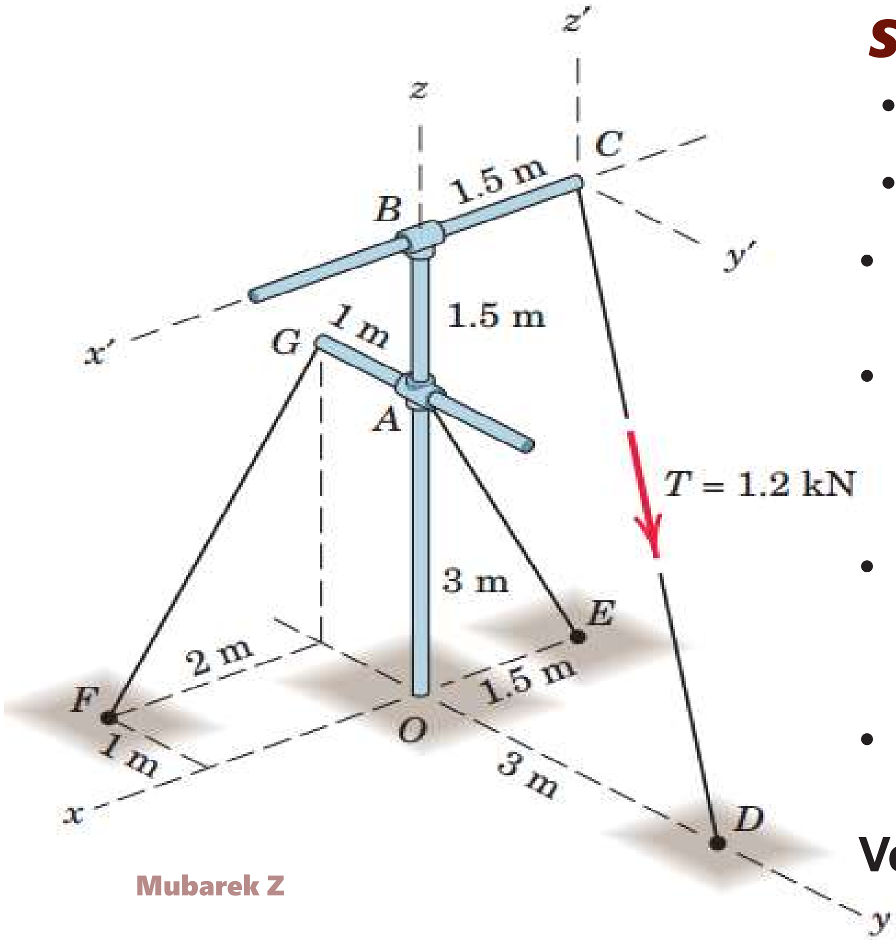


Figure 2/18

we must use a right-handed set of axes in our three-dimensional work to be consistent with the right-hand-rule definition of the cross product. When we rotate from the x- to the y-axis through the 90° angle, the positive direction for the z-axis in a right-handed system is that of the advancement of a right-handed screw rotated in the same sense. This is equivalent to the right-hand rule

3D System …

Example: The rigid pole and cross-arm assembly is supported by the three cables shown. A turnbuckle at D is tightened until it induces a tension T in CD of 1.2 kN. Express T as a vector. Does it make any difference in the result which coordinate system is used?



Solution:

- **Coordinate of point C(-1.5,0,4.5)**
- **Coordinate of point D(0,3,0)**
- **Vector CD=(1.5,3,-4.5) or**
  CD=1.5i + 3j-4.5k and
- **mag. of CD**
  /CD/=$\sqrt{1.5*1.5+3*3+(-4.5)*(-4.5)}$
  /CD/=$\sqrt{31}.5$**=5.61**
- **Unit vector of CD, n_{CD}=CD/(/CD/)**
  n_{CD}=CD/(/CD/)=[1.5i + 3j-4.5k]/5.61
  n_{CD}=0.267 i + 0.535 j – 0.802 k
- **Vector, T=/T/ * n_{CD}**
  T=1.2*[0.267 i + 0.535 j – 0.802 k] kN

Vector T= 0.321 i + 0.641 j – 0.962 k kN,
NO, co. system do not affect. (Ans)

3D System …

Moment in 3D

- In 2D analyses it is often convenient to determine a moment magnitude by scalar multiplication using the moment-arm rule. In 3Ds, however, the determination of the perpendicular distance between a point or line and the line of action of the force can be a tedious computation. A vector approach with cross-product multiplication then becomes advantageous

Evaluating the Cross Product

The cross-product expression for M_O may be written in the determinant form

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

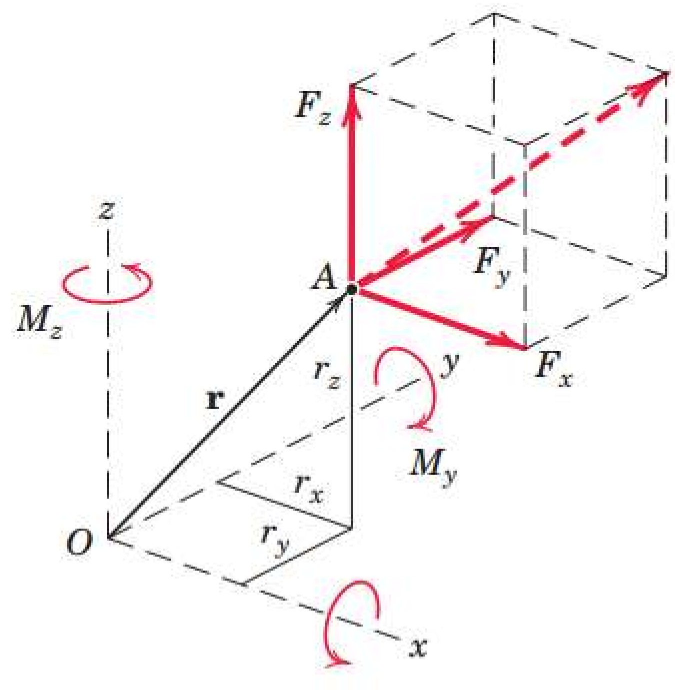


Figure 2/22

Note the symmetry and order of the terms, and note that a right-handed coordinate system must be used. Expansion of the determinant gives

$$\mathbf{M}_O = (r_y F_z - r_z F_y)\mathbf{i} + (r_z F_x - r_x F_z)\mathbf{j} + (r_x F_y - r_y F$$

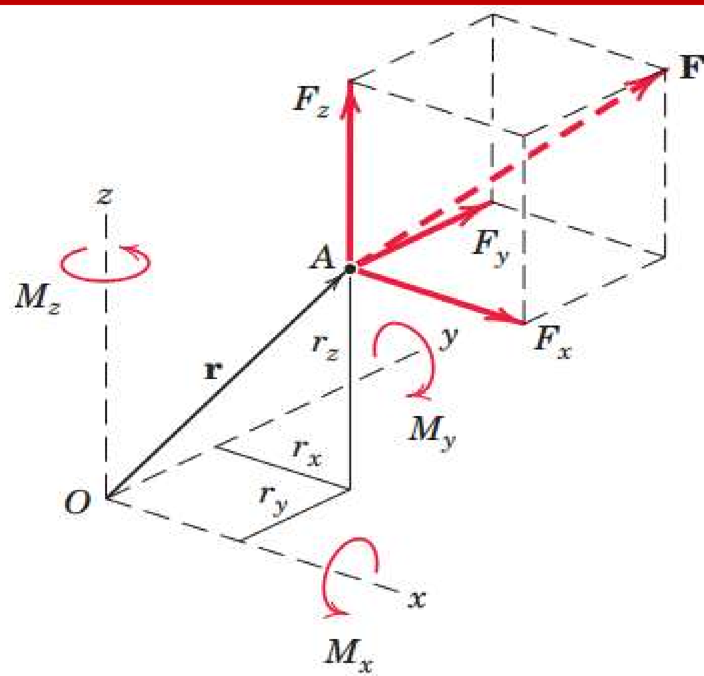
3D System …

Moment in 3Ds …

♠ *The scalar magnitudes of the moments of these forces about the positive x-, y-, and z-axes through O can be obtained from the moment-arm rule, and are* $M_x = r_y F_z - r_z F_y \qquad M_y = r_z F_x - r_x F_z \qquad M_z = r_x F_y - r_y F_x$

which agree with the respective terms in the determinant expansion for the cross product, rXF.

♠ *The magnitude of the moment, /M/=*$\sqrt{(Mx^2 + My^2 + Mz^2)}$



3D System …

Varignon's Theorem in Three Dimensions

♠ *The theorem is easily extended to three dimensions. Figure 2/24 shows a system of concurrent forces F1, F2, F3, ... The sum of the moments about O of these forces is*

$$\mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 + \mathbf{r} \times \mathbf{F}_3 + \cdots = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots)$$
$$= \mathbf{r} \times \Sigma\mathbf{F}$$

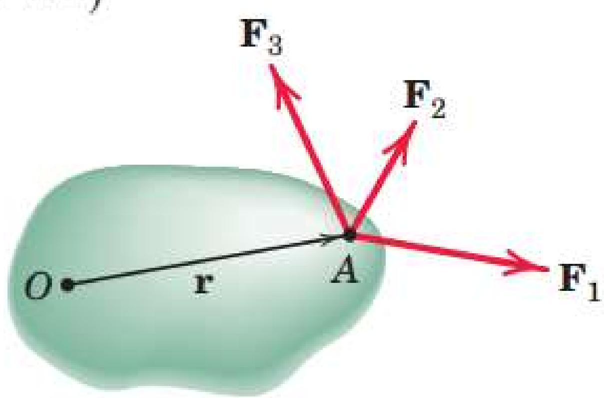


Figure 2/24

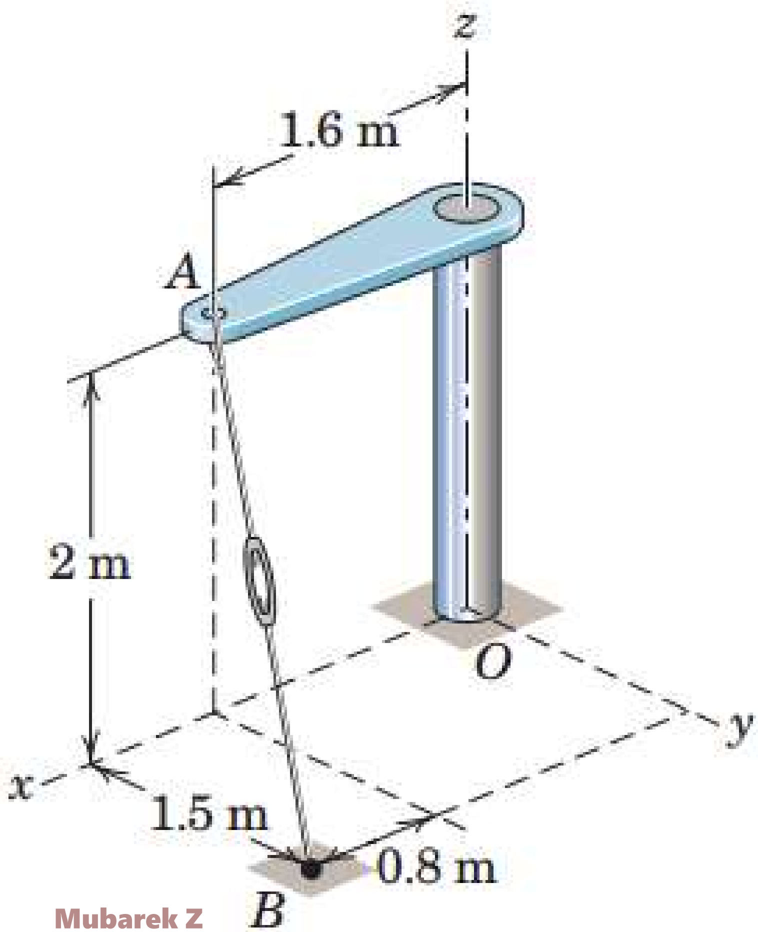
where we have used the distributive law for cross products. Using the symbol M_O to represent the sum of the moments on the left side of the above equation, we have

$$\mathbf{M}_O = \Sigma(\mathbf{r} \times \mathbf{F}) = \mathbf{r} \times \mathbf{R}$$

This equation states that the sum of the moments of a system of concurrent forces about a given point equals the moment of their sum about the same point.

3D System …

Example: The turnbuckle is tightened until the tension in cable AB is 2.4 kN. Determine the moment about point O of the cable force acting on point A and the magnitude of this moment.



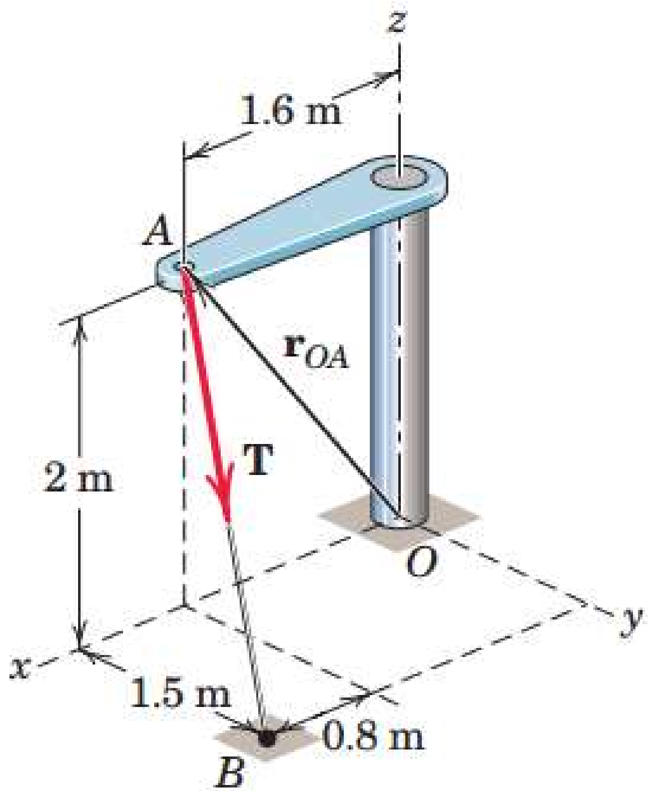
Mubarek Z

Solution:

- ✓ ***First determine coordinates of necessary points, in this case point O, A, and B.***
- ✓ ***Write the force as a vector***
- ✓ ***Determine position vector, r_{OA}***
- ✓ ***Determine moment about O, Use the cross product as M=rXF***

3D System …

Solution: **We begin by writing the described force as a vector.**



- **Coordinate of point A(1.6,0,2) & of point B(2.4,1.5,0)**
- **Vector AB=(0.8,1.5,-2) and**
- **Mag. Of AB,**

/AB/=$\sqrt{0.8*0.8+1.5*1.5+(-2)*(-2)}$

/AB/=$\sqrt{(6.89)}$**=2.625**

- **Unit vector of AB, n_{AB}=AB/(/AB/)**
- **n_{AB}=[0.8i + 1.5j-2k]/2.625**

Vectorial representation of tension force, T

- Vector, T=/T/ * n_{AB}

T=2.4*[0.3047 i + 0.571 j – 0.762 k] kN

Vector T= 0.731 i + 1.371 j – 1.829 k (kN),

- **Moment arm, r_{OA}=?**
- **Coordinate of O (0,0,0) &Coordinate of A(1.6,0,2)**
- **Vector OA= r_{OA}=(1.6,0,2)=1.6 i+0 j+ 2 k (m)**
- ✓ **Moment about O, M_O=r_{OA}XT**

$$\mathbf{M}_O = \mathbf{r}_{OA} \times \mathbf{T} = (1.6\mathbf{i} + 2\mathbf{k}) \times (0.731\mathbf{i} + 1.371\mathbf{j} - 1.829\mathbf{k}) = -2.74\mathbf{i} + 4.39\mathbf{j} + 2.19\mathbf{k}\ \text{kN}\cdot\text{m}$$

This vector has a magnitude, /Mo/ $= \sqrt{2.74^2 + 4.39^2 + 2.19^2} = 5.62\ \text{kN}\cdot\text{m}$

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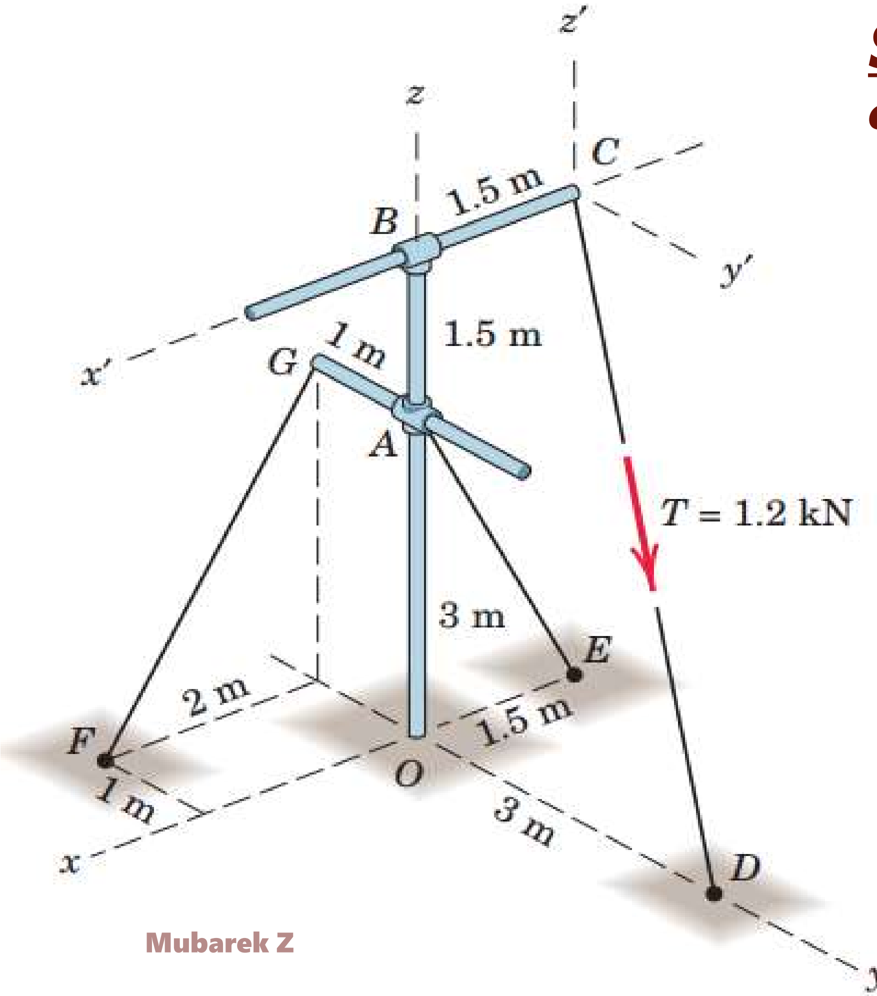
(Ans)

3D System …

Example: The rigid pole and cross-arm assembly of example done previously is shown again here. Determine the vector expression for the moment of the 1.2-kN tension (a) about point O and (b) about the pole z-axis (c) about point B



Solution: ***Vector T is the same for all cases but the moment arm is different***

- Coordinate of point C(-1.5,0,4.5)
- Coordinate of point D(0,3,0)
- Vector CD=(1.5,3,-4.5) or
  CD=1.5i + 3j-4.5k and
- mag. of CD
  /CD/=$\sqrt{1.5*1.5+3*3+(-4.5)*(-4.5)}$
  /CD/=$\sqrt{31.5}$=5.61
- Unit vector of CD, n_{CD}=CD/(/CD/)
  n_{CD}=CD/(/CD/)=[1.5i + 3j-4.5k]/5.61
  n_{CD}=0.267 i + 0.535 j – 0.802 k
- Vector, T=/T/ * n_{CD}
  T=1.2*[0.267 i + 0.535 j – 0.802 k] kN

Vector T= 0.321 i + 0.641 j – 0.962 k kN,

3D System …

Solution… **(a) about point O**

- **Moment arm, r_{OC}=? Using the Figure Coordinate of point O (0,0,0) & that of point C(-1.5,0,4.5)**
- **Vector OC= r_{OC}=(-1.5,0,4.5)= -1.5 i+0 j+ 4.5 k (m)**
- ✓ **Moment about O, $M_O=r_{OC}XT$**

M_O=[-1.5 i+0 j+ 4.5 k]X[0.321 i + 0.641 j – 0.962 k]; $Mo = \begin{vmatrix} i & j & k \\ -1.5 & 0 & 4.5 \\ 0.321 & 0.641 & -0.962 \end{vmatrix}$

Mo=[(0∗(−0.962)−(0.641∗4.5)] i −[((−1.5)∗(−0.962))−(0.321∗4.5)] j+

[((-1.5)*0.641)-(0.321*0)]k=-2.885 i+0.0j -0.962 k (kNm) …..(Ans)

(b) about point Z-axis, M_Z=-0.962 k (kNm) **z-component of Mo**

(c) about point B, M_B=?

- **Moment arm, r_{BC}=?**
- **Coordinate of point B (0,0,4.5) & of point C(-1.5,0,4.5)**
- **Vector BC= r_{BC}=(-1.5,0,0)= -1.5 i+0 j+ 0 k =-1.5 i (m)**
- ✓ **Moment about B, $M_B=r_{BC}XT$**

$Mo = \begin{vmatrix} i & j & k \\ -1.5 & 0 & 0 \\ 0.321 & 0.641 & -0.962 \end{vmatrix}$

M_B=[-1.5 i]X[0.321 i + 0.641 j – 0.962 k] ;

M_B=[(0∗(−0.962)−(0.641∗0)] i −[((−1.5)∗(−0.962))−(0.321∗0)] j+

[((-1.5)*0.641)-(0.321*0)]k= -1.443 j -0.962 k (kNm) …..(Ans) (Ans)

3D System …

Couples in Three Dimensions

- A couple is the combined moment of two equal, opposite, and non-collinear forces.
- The unique effect of a couple is to produce a pure twist or rotation regardless of where the forces are located.
- Figure 2/25 shows two equal and opposite forces F and -F acting on a body. The vector r runs from *any* point *B* on the line of action of -F to *any* point *A* on the line of action of F. Points *A* and *B* are located by position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ from *any* point *O*. The combined moment of the two forces about *O* is

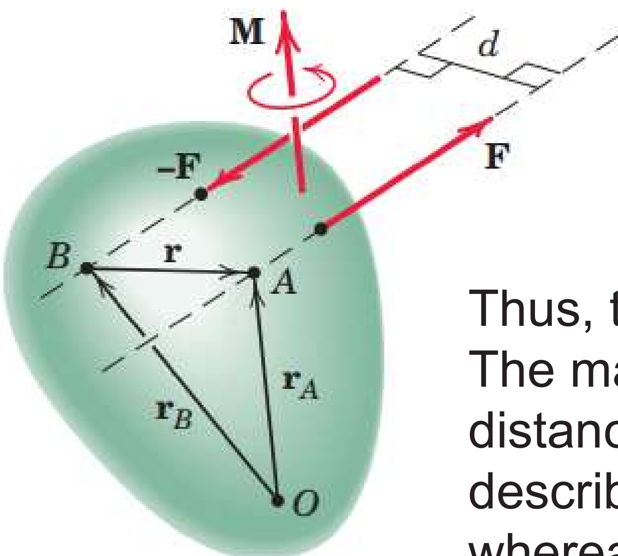


Figure 2/25

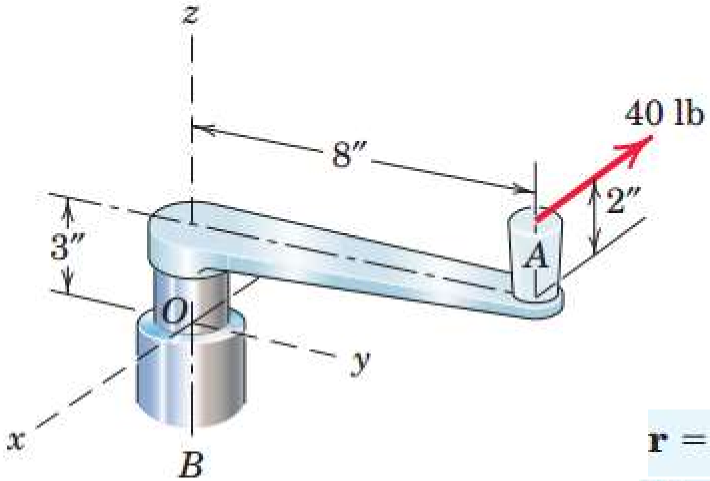
$$\mathbf{M} = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}$$

However, $\mathbf{r}_A - \mathbf{r}_B = \mathbf{r}$, so that all reference to the moment center *O* disappears, and the moment of the couple becomes $\mathbf{M} = \mathbf{r} \times \mathbf{F}$

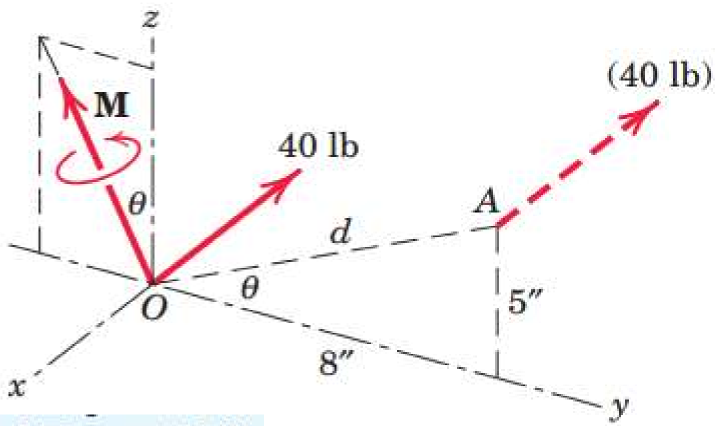
Thus, the moment of a couple is the *same about all points*. The magnitude of **M** is $M = Fd$, where *d* is the perpendicular distance between the lines of action of the two forces, as described above. The moment of a couple is a *free vector*, whereas the moment of a force about a point (which is also the moment about a defined axis through the point) is a *sliding vector* whose direction is along the axis through the point

3D System …

Example: A force of 40 lb is applied at A to the handle of the control lever which is attached to the fixed shaft OB. In determining the effect of the force on the shaft at a cross section such as that at O, we may replace the force by an equivalent force at O and a couple. Describe this couple as a vector M.



Solution



$\mathbf{r} = \overrightarrow{OA} = 8\mathbf{j} + 5\mathbf{k}$ in. and $\mathbf{F} = -40\mathbf{i}$ lb.

$\mathbf{M} = \mathbf{r} \times \mathbf{F}$, $\mathbf{M} = (8\mathbf{j} + 5\mathbf{k}) \times (-40\mathbf{i}) = -200\mathbf{j} + 320\mathbf{k}$ lb-in.

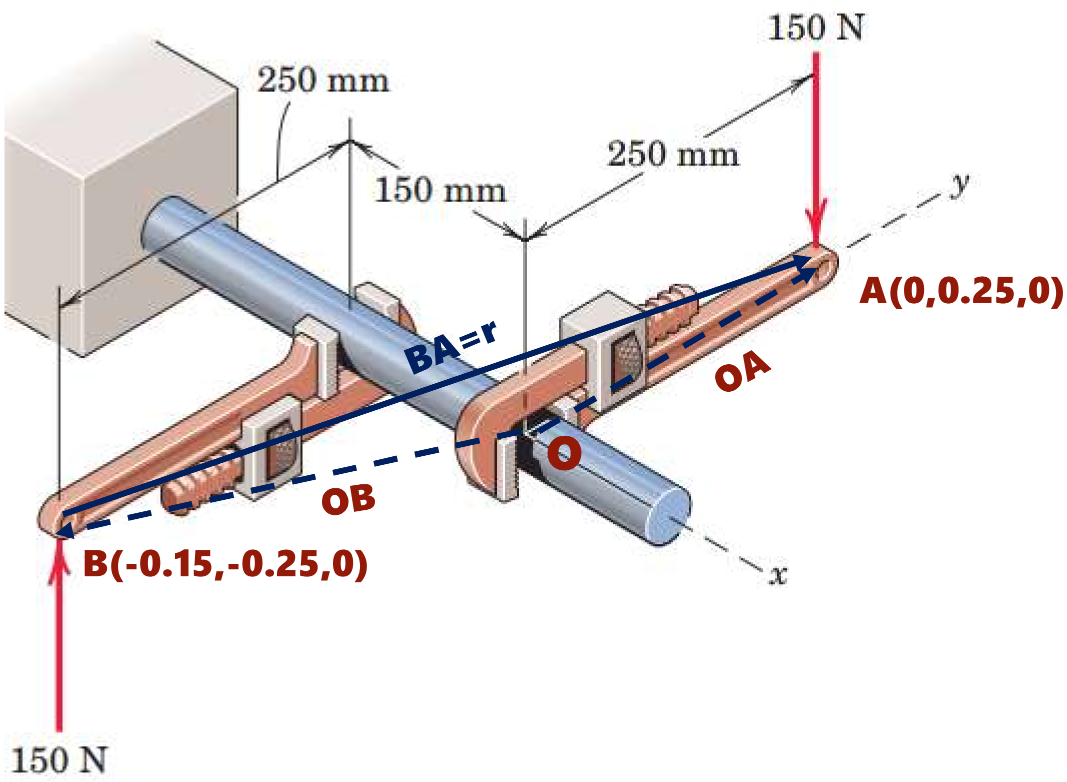
Alternatively we see that moving the 40-lb force through a distance $d = \sqrt{5^2 + 8^2} = 9.43$ in. to a parallel position through O requires the addition of a couple $\mathbf{M}$ whose magnitude is

$$M = Fd = 40(9.43) = 377 \text{ lb-in.} \qquad Ans.$$

The couple vector is perpendicular to the plane in which the force is shifted, and its sense is that of the moment of the given force about *O*. The direction of **M** in the *y*-*z* plane is given by $\theta = \tan^{-1}\frac{5}{8} = 32.0°$

3D System …

Exercise: The two forces acting on the handles of the pipe wrenches constitute a couple M. Express the couple as a vector.



r=BA=0.15i +0.5j+0k
F=-150k (N)
Mo=rXF

$$Mo = \begin{vmatrix} i & j & k \\ 0.15 & 0.5 & 0 \\ 0 & 0 & -150 \end{vmatrix}$$

=[0.5*(-150)-0] i-[0.15*(-150)-0] j+0k

Mo=-75 i + 22.5 j (Nm)….Answer

3D System …

Resultant

- *We defined the resultant as the simplest force combination which can replace a given system of forces without altering the external effect on the rigid body on which the forces act. For example, for the Force system* $\boldsymbol{F_1}$, $\boldsymbol{F_2}$, $\boldsymbol{F_3}$ *. . . acting on a rigid body shown in Figure (a). we may move each of them in turn to the arbitrary point O, provided we also introduce a couple for each force transferred. Thus, for example, we may move force* F_1 *to O, provided we introduce the couple* $M_1 = r_1 X F_1$*, where* r_1 *is a vector from O to any point on the line of action of* F_1*. When all forces are shifted to O in this manner, we have a system of concurrent forces at O and a system of couple vectors as in Figure (b). The concurrent forces added vectorially to produce a resultant force R, and the couples may also be added to produce a resultant couple M*

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \Sigma \mathbf{F}$$

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \cdots = \Sigma(\mathbf{r} \times \mathbf{F})$$

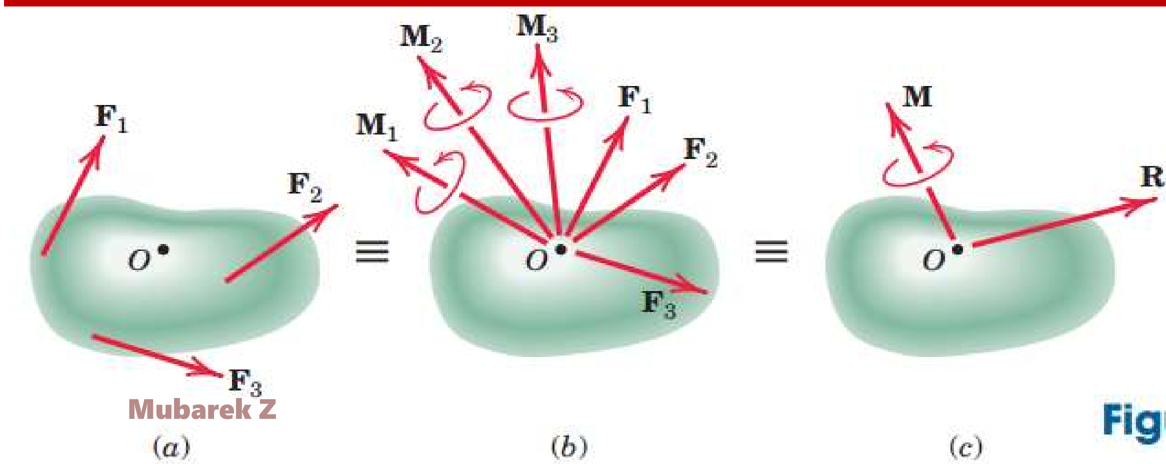


Figure 2/28

3D System …

Resultant

- *The couple vectors are shown through point O, but because they are free vectors, they may be represented in any parallel positions. The magnitudes of the resultants and their components are*

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R_z = \Sigma F_z$$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$

$$\mathbf{M}_x = \Sigma(\mathbf{r} \times \mathbf{F})_x \qquad \mathbf{M}_y = \Sigma(\mathbf{r} \times \mathbf{F})_y \qquad \mathbf{M}_z = \Sigma(\mathbf{r} \times \mathbf{F})_z$$

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$

- *The point O selected as the point of concurrency for the forces is arbitrary, and the magnitude and direction of M depend on the particular point O selected. The magnitude and direction of R, however, are the same no matter which point is selected.*

Parallel Forces: For a system of parallel forces not all in the same plane, the magnitude of the parallel resultant force R is simply the magnitude of the algebraic sum of the given forces. The position of its line of action is obtained from the principle of moments by requiring that r XR = M_O. Here r is a position vector extending from the force–couple reference point O to the final line of action of R, and M_O is the sum of the moments of the individual forces about O.

3D System …

Resultant

- ✓ *Wrench Resultant:* *When the resultant couple vector M is parallel to the resultant force R, as shown in Fig. 2/29, the resultant is called a wrench.*
- ✓ *By definition a wrench is positive if the couple and force vectors point in the same direction and negative if they point in opposite directions.*
- ✓ *A common example of a positive wrench is found with the application of a screwdriver, to drive a right-handed screw.*

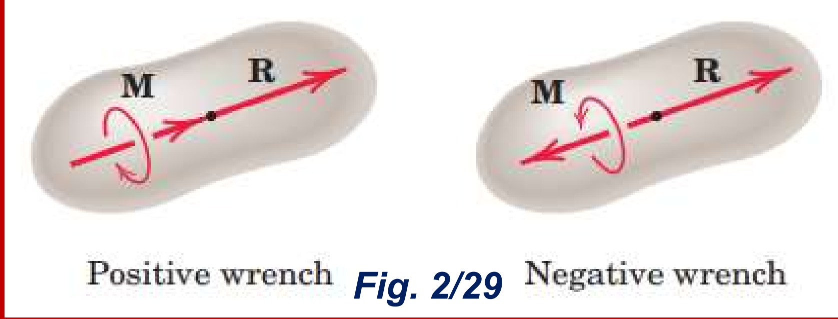


Fig. 2/29

- ✓ *Any general force system as shown in Figure 2/30 (a) below may be represented by a wrench applied along a unique line of action. This reduction is illustrated in Figure 2/30*

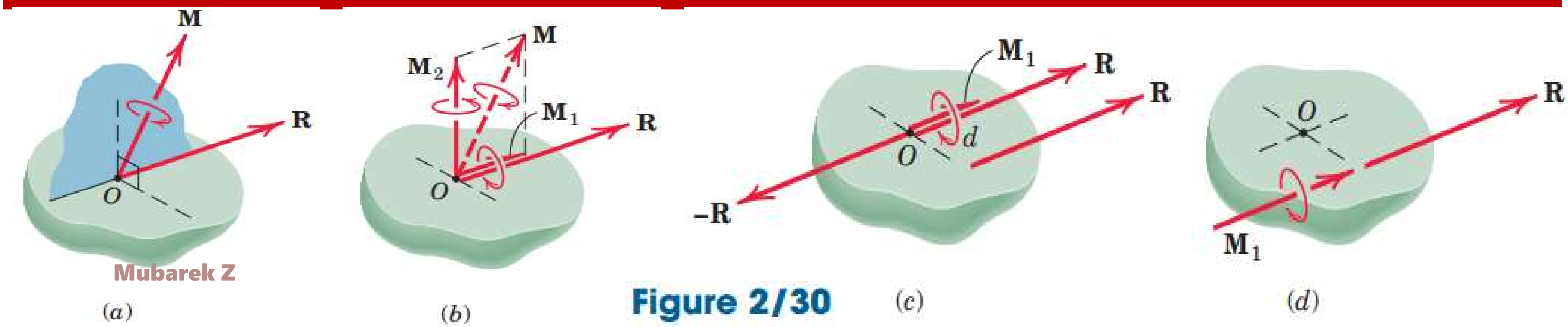
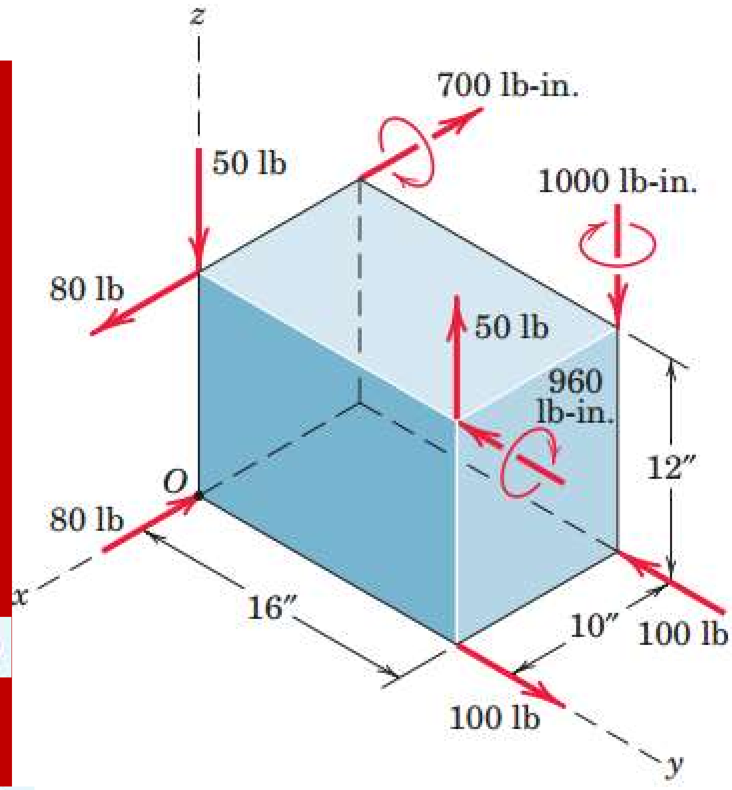


Figure 2/30

3D System …



Example 1

1. Determine the resultant of the force and couple system which acts on the rectangular solid.

Solution: *We choose point O as a convenient reference point for the initial step of reducing the given forces to a force–couple system.*

The resultant force is

$$\mathbf{R} = \Sigma\mathbf{F} = (80 - 80)\mathbf{i} + (100 - 100)\mathbf{j} + (50 - 50)\mathbf{k} = \mathbf{0} \text{ lb}$$

The sum of the moments about O is

$$\mathbf{M}_O = [50(16) - 700]\mathbf{i} + [80(12) - 960]\mathbf{j} + [100(10) - 1000]\mathbf{k} \text{ lb-in.}$$
$$= 100\mathbf{i} \text{ lb-in.}$$

N.B

- ✓ **Since the force summation is zero, we conclude that the resultant, if it exists, must be a couple.**
- ✓ **The moments associated with the force pairs are easily obtained by using the M =F*d rule and assigning the unit-vector direction by inspection. In many three-dimensional problems, this may be simpler than the M = r XF approach.**

3D System …

Example 2

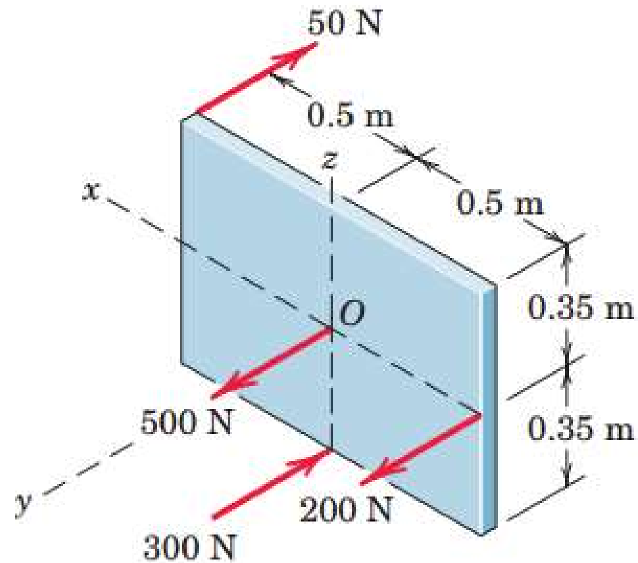
2. Determine the resultant of the system of parallel forces which act on the plate. Solve with a vector approach.

<u>Solution:</u> Transfer of all forces to point O results in the force–couple system

- ***The resultant force R and Couple M about point O***

$$\mathbf{R} = \Sigma\mathbf{F} = (200 + 500 - 300 - 50)\mathbf{j} = 350\mathbf{j}\ \text{N}$$

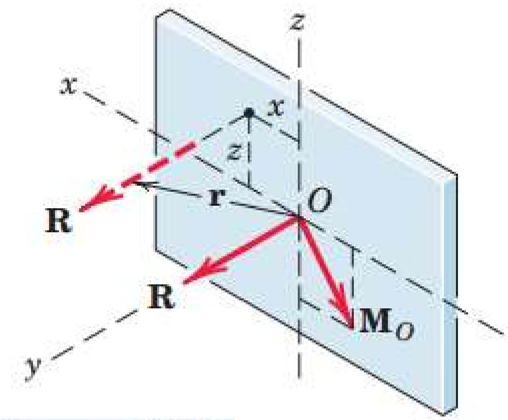
$$\mathbf{M}_O = [50(0.35) - 300(0.35)]\mathbf{i} + [-50(0.50) - 200(0.50)]\mathbf{k} = -87.5\mathbf{i} - 125\mathbf{k}\ \text{N}\cdot\text{m}$$



The placement of R so that it alone represents the above force–couple system is determined by the principle of moments in vector form $\mathbf{r} \times \mathbf{R} = \mathbf{M}_O$

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times 350\mathbf{j} = -87.5\mathbf{i} - 125\mathbf{k}$$

$$350x\mathbf{k} - 350z\mathbf{i} = -87.5\mathbf{i} - 125\mathbf{k}$$



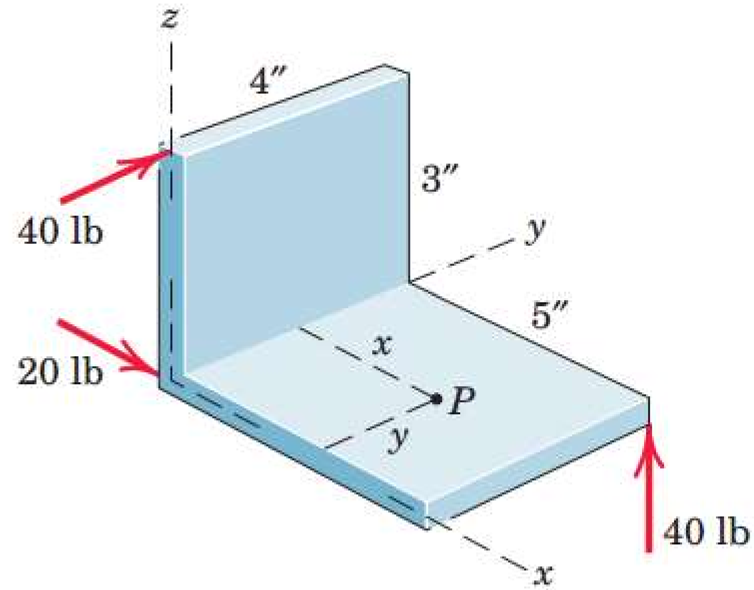
From the equality of vectors we get $350x = -125 \quad \text{and} \quad -350z = -87.5$

Solving gives x=-0.357 m and z = 0.250 m are the coordinates through which the line of action of R must pass. The value of y can be any value, as permitted by the principle of transmissibility. Thus, as expected, the variable y drops out of the above vector analysis.

3D System …

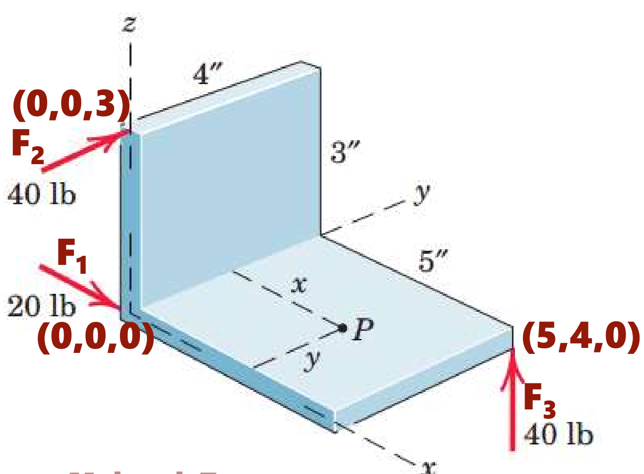
Example 3

1. Determine the wrench resultant of the three forces acting on the bracket. Calculate the coordinates of the point P in the x-y plane through which the resultant force of the wrench acts. Also find the magnitude of the couple M of the wrench.



Solution:

- **Calculate the resultant at origin O(0,0,0)**



- **Writing Forces as a vector**

F_1=20 i (lb) F_2=40 j (lb) F_3=40 k (lb)

Resultant force R= F_1+F_2+F_3

R=20i + 40j +40k (lb)

/R/=60 lb

- **Writing position vector of the moment arm to the point of application of the forces**

r_1 =0, r_2 =3k (in), r_2 =5i +4j (in)

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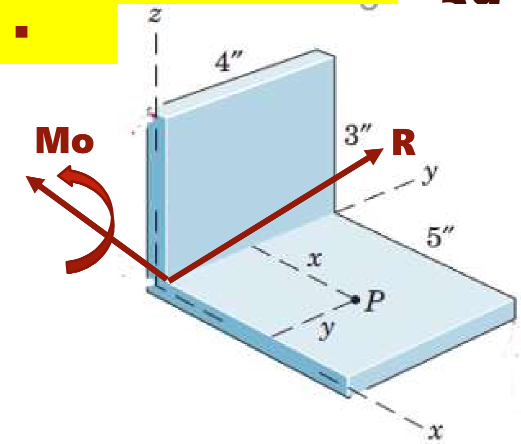
2. FORCE SYSTEMS…

3D System …

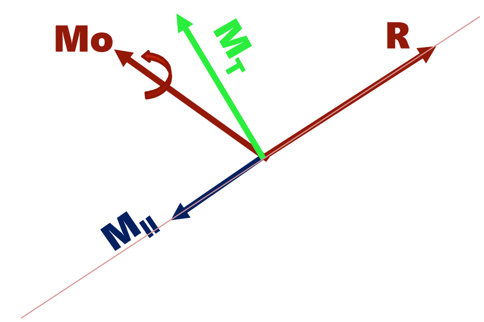
- $Mo = r_1 x F_1 + r_2 X F_2 + r_3 X F_3$

$$Mo = \begin{vmatrix} i & j & k \\ 0 & 0 & 0 \\ 20 & 0 & 0 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & 3 \\ 0 & 40 & 0 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 5 & 4 & 0 \\ 0 & 0 & 40 \end{vmatrix}$$

$$\mathbf{Mo} = 0 - 120i + 160i - 200j = 40i - 200j \text{ (lbin)}$$



Force-couple system



- **Convert the Force-couple system to the wrench system, determine the parallel projection of the moment(M_{II}) in the direction of R. & then the perpendicular projection(M_T) of the moment Mo.**

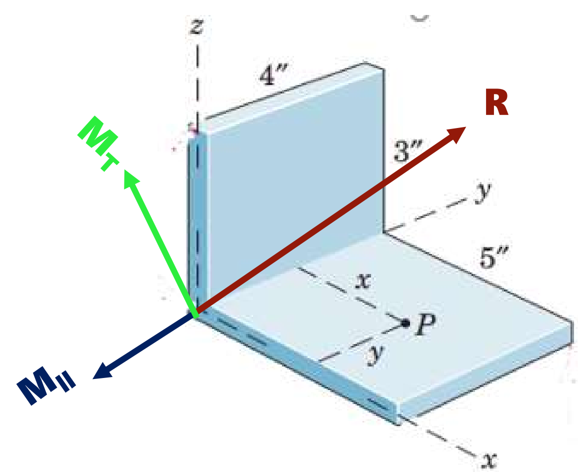
$$M_{II} = \frac{Mo\ .\ R}{/R/2}\ \mathbf{R}$$

Mo .R=40*20+(-200*40)+0*40= -7200

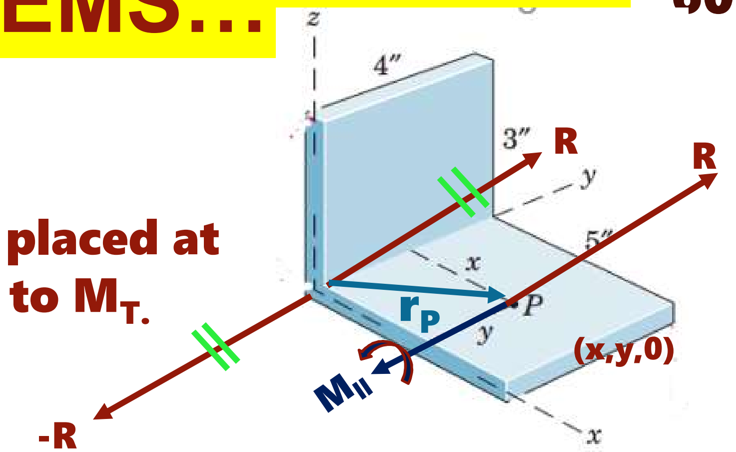
$$M_{II} = \frac{Mo\ .\ R}{/R/2}\ \mathbf{R} = \frac{-7200}{60*60}*[20i + 40j + 40k], \quad M_{II} = -40i - 80j - 80k$$

Thus $M_T = M - M_{II}$; $M_T = [40i - 200j] - [-40i - 80j - 80k]$

$M_T = 80i - 120j + 80k$



3D System …



- ✓ To ignore M_T, the resultant force R should be placed at P(x,y,0) so that it make a couple that is equal to M_T.
- ✓ Thus, $r_P X R = M_T$ where r_P= xi +yj +0k

$$\begin{vmatrix} i & j & k \\ x & y & 0 \\ 20 & 40 & 40 \end{vmatrix} = 80i - 120j + 80k$$

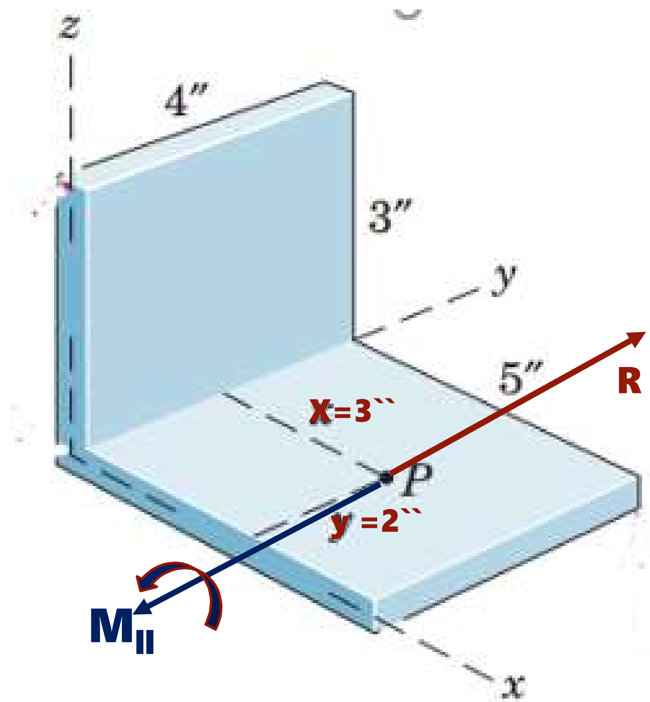
❖ 40yi -40xj +[40x-20y]k=80i -120j +80k

Solve 40y=80, y=2 in and -40x=-120, x=3 in

- R be locate at P(3,2,0)….Ans.
- Magnitude of the couple of the wrench,/M_{II}/

/M_{II}/=$\sqrt{(-40 * -40) + (-80 * -80) + (-80 * -80)}$

/MII/=120 lbin…Ans.

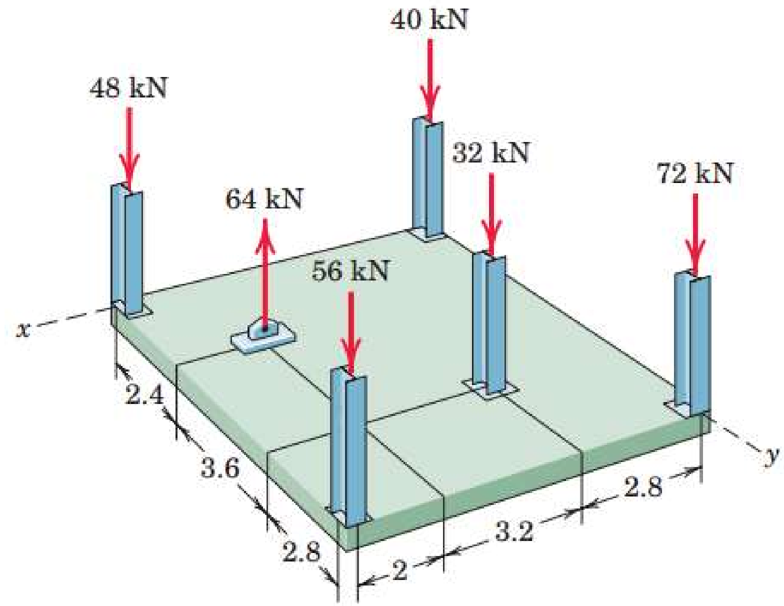


Wrench resultant(-ve wrench) …Ans.

3D System …

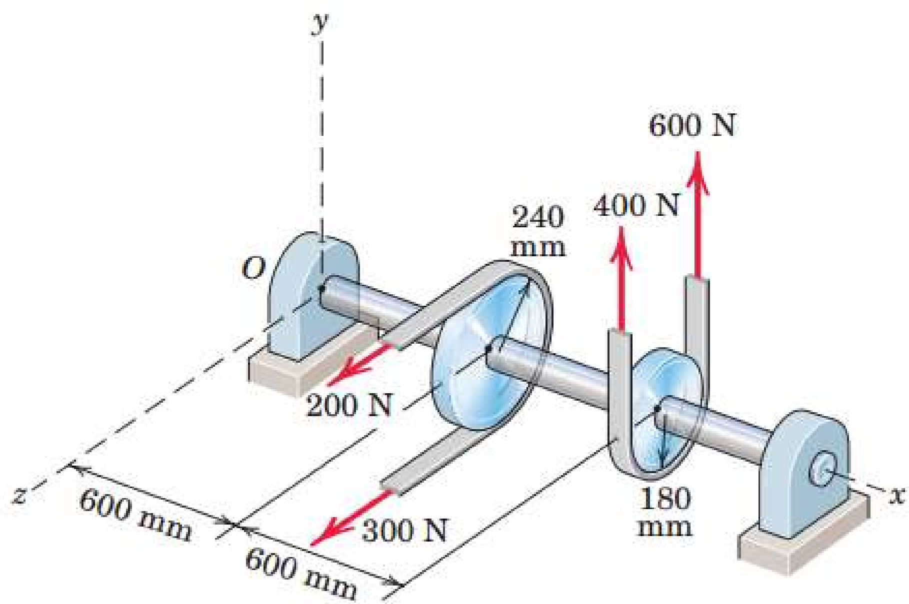
Exercise

1) ***The concrete slab supports the six vertical loads shown. Determine the x- and y-coordinates of the point on the slab through which the resultant of the loading system passes.***



Dimensions in meters

2) ***The pulley wheels are subjected to the loads shown. Determine the equivalent force–couple system at point O.***



3. EQUILIBRIUM

Introduction

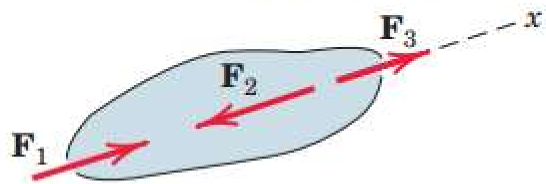
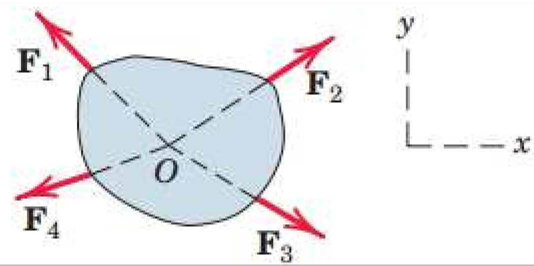
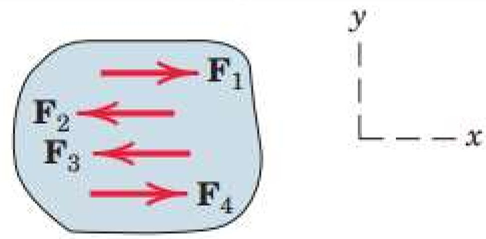
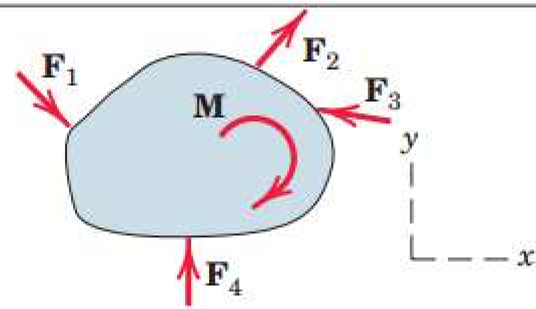
- ✓ Equilibrium is a condition in which all influences acting cancel each other, so that a static or balanced situation results.
- ✓ When a body is in equilibrium, the resultant of all forces acting on it is zero. Thus, the resultant force R and the resultant couple M are both zero, and we have the equilibrium equations $\mathbf{R} = \Sigma\mathbf{F} = 0 \qquad \mathbf{M} = \Sigma\mathbf{M} = 0$
- ♠ These requirements are both necessary and sufficient conditions for equilibrium.

3. EQUILIBRIUM…

Section A Equilibrium in 2Ds

Equilibrium Conditions

We defined equilibrium as the condition in which the resultant of all forces and moments acting on a body is zero. Stated in another way, a body is in equilibrium if all forces and moments applied to it are in balance.

CATEGORIES OF EQUILIBRIUM IN TWO DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Collinear		$\Sigma F_x = 0$
2. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$
3. Parallel		$\Sigma F_x = 0 \quad \Sigma M_z = 0$
4. General		$\Sigma F_x = 0 \quad \Sigma M_z = 0$ $\Sigma F_y = 0$

SECTION A EQUILIBRIUM IN TWO DIMENSIONS

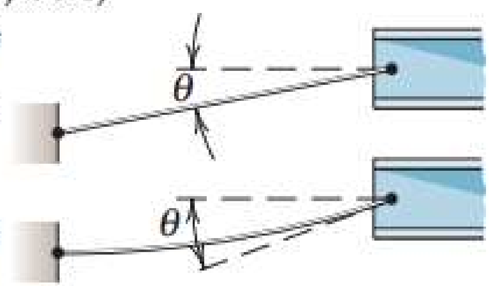
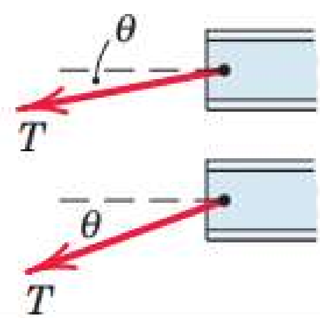
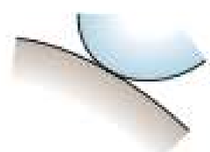
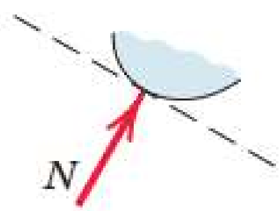
System Isolation and the Free-Body Diagram

- ✓ Before we apply the above equilibrium equations we must define unambiguously the particular body or mechanical system to be analyzed and represent clearly and completely all forces acting on the body. Omission of a force which acts on the body in question, or inclusion of a force which does not act on the body, will give erroneous results.
- ✓ Once we decide which body or combination of bodies to analyze, we then treat this body or combination as a single body isolated from all surrounding bodies. This isolation is accomplished by means of the free-body diagram, which is a diagrammatic representation of the isolated system treated as a single body. The diagram shows all forces applied to the system by mechanical contact with other bodies, which are imagined to be removed.

SECTION A EQUILIBRIUM IN TWO DIMENSIONS

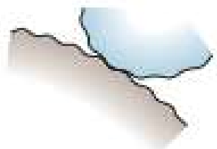
Modeling the Action of Forces

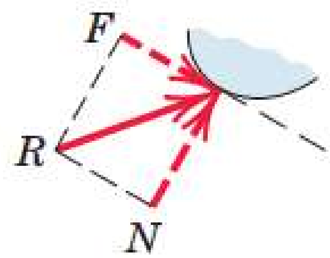
- **The Figure below shows the common types of force application on mechanical systems for analysis in 2D. Each example shows the force exerted on the body to be isolated, by the body to be removed. Newton's third law, which notes the existence of an equal and opposite reaction to every action, must be carefully observed.**

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.

SECTION A EQUILIBRIUM IN TWO DIMENSIONS

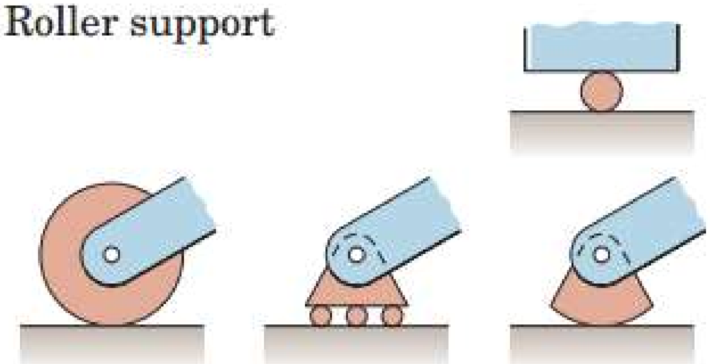
3. Rough surfaces

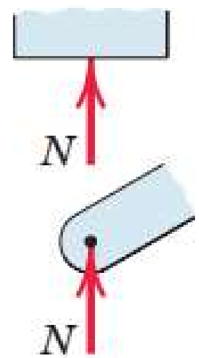




Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.

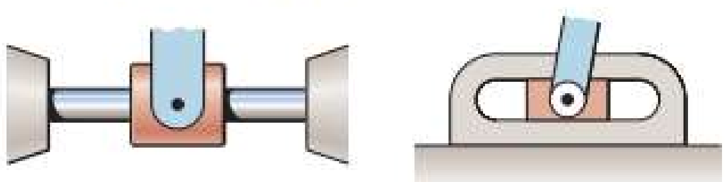
4. Roller support

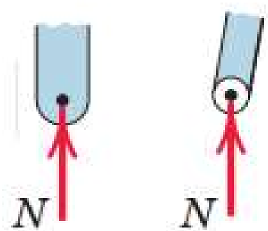




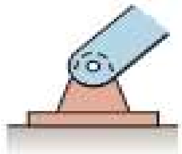
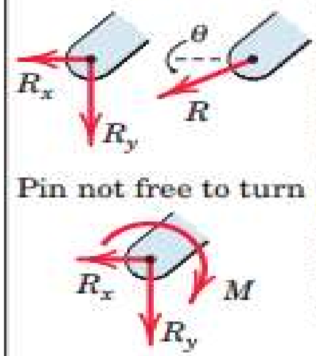
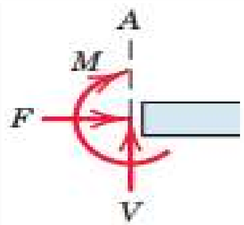
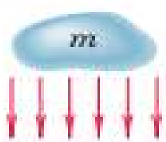
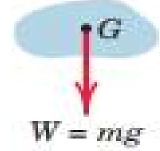
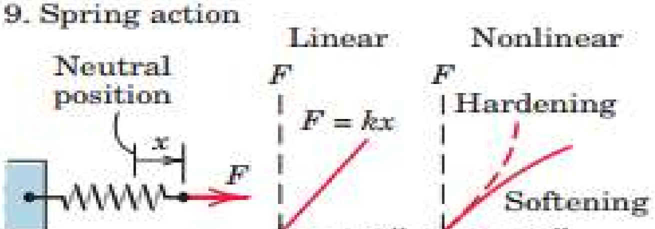
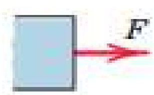
Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.

5. Freely sliding guide





Collar or slider free to move along smooth guides; can support force normal to guide only.

6. Pin connection 		A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.
7. Built-in or fixed support 		A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction 		The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.
9. Spring action 		Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

Section A Equilibrium in 2Ds

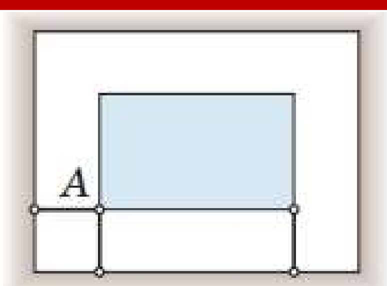
Constraints and Statical Determinacy

- ✓ The equilibrium equations developed in this chapter are both necessary and sufficient conditions to establish the equilibrium of a body. However, they do not necessarily provide all the information required to calculate all the unknown forces which may act on a body in equilibrium. Whether the equations are adequate to determine all the unknowns depends on the characteristics of the constraints against possible movement of the body provided by its supports.
- ✓ By constraint we mean the restriction of movement. Example:- the roller, ball, and rocker provide constraint normal to the surface of contact, but none tangent to the surface.
- ✓ A rigid body, or rigid combination of elements treated as a single body, which possesses more external supports or constraints than are necessary to maintain an equilibrium position is called statically indeterminate.
- ✓ Supports which can be removed without destroying the equilibrium condition of the body are said to be redundant.
- ✓ The number of redundant supporting elements present corresponds to the degree of statical indeterminacy and equals the total number of unknown external forces, minus the number of available independent equations of equilibrium. On the other hand, bodies which are supported by the minimum number of constraints necessary to ensure an equilibrium configuration are called statically determinate, and for such bodies the equilibrium equations are sufficient to determine the unknown external forces.

Section A Equilibrium in 2Ds

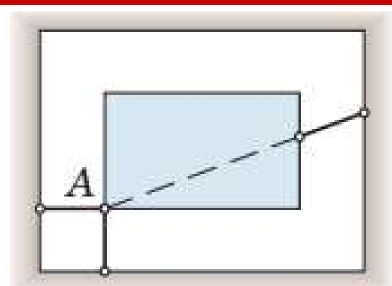
Adequacy of Constraints

- ✓ **We must be aware of the nature of the constraints before we attempt to solve an equilibrium problem. The existence of three constraints for a two-dimensional problem does not always guarantee a stable equilibrium configuration.**
- ✓ **In Figure (a), at point A of the rigid body is fixed by the two links and cannot move, and the third link prevents any rotation about A. Thus, this body is completely fixed with three adequate (proper) constraints.**
- ✓ **In Figure (b), there is no resistance to rotation at A where as in Figure (c) the three parallel links could offer no initial resistance to a small vertical movement of the body as a result of external loads applied to it in this direction. The constraints in these two examples are often termed improper. In both cases, this body is incompletely fixed under partial constraints.**
- ✓ **In Figure (d), we have a condition of complete fixity, with link 4 acting as a fourth constraint which is unnecessary to maintain a fixed position. Link 4, then, is a redundant constraint, and the body is statically indeterminate.**

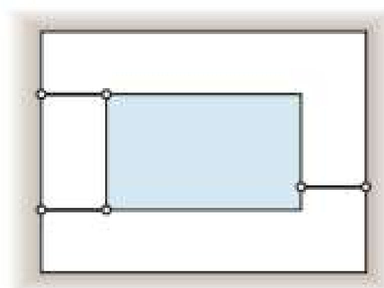


(*a*) Complete fixity

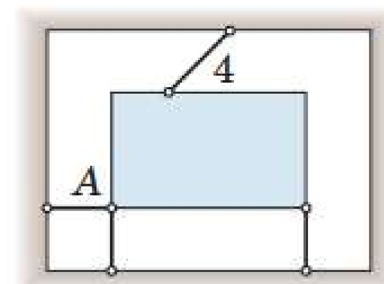
Adequate constraints



(*b*) Incomplete fixity
Partial constraints



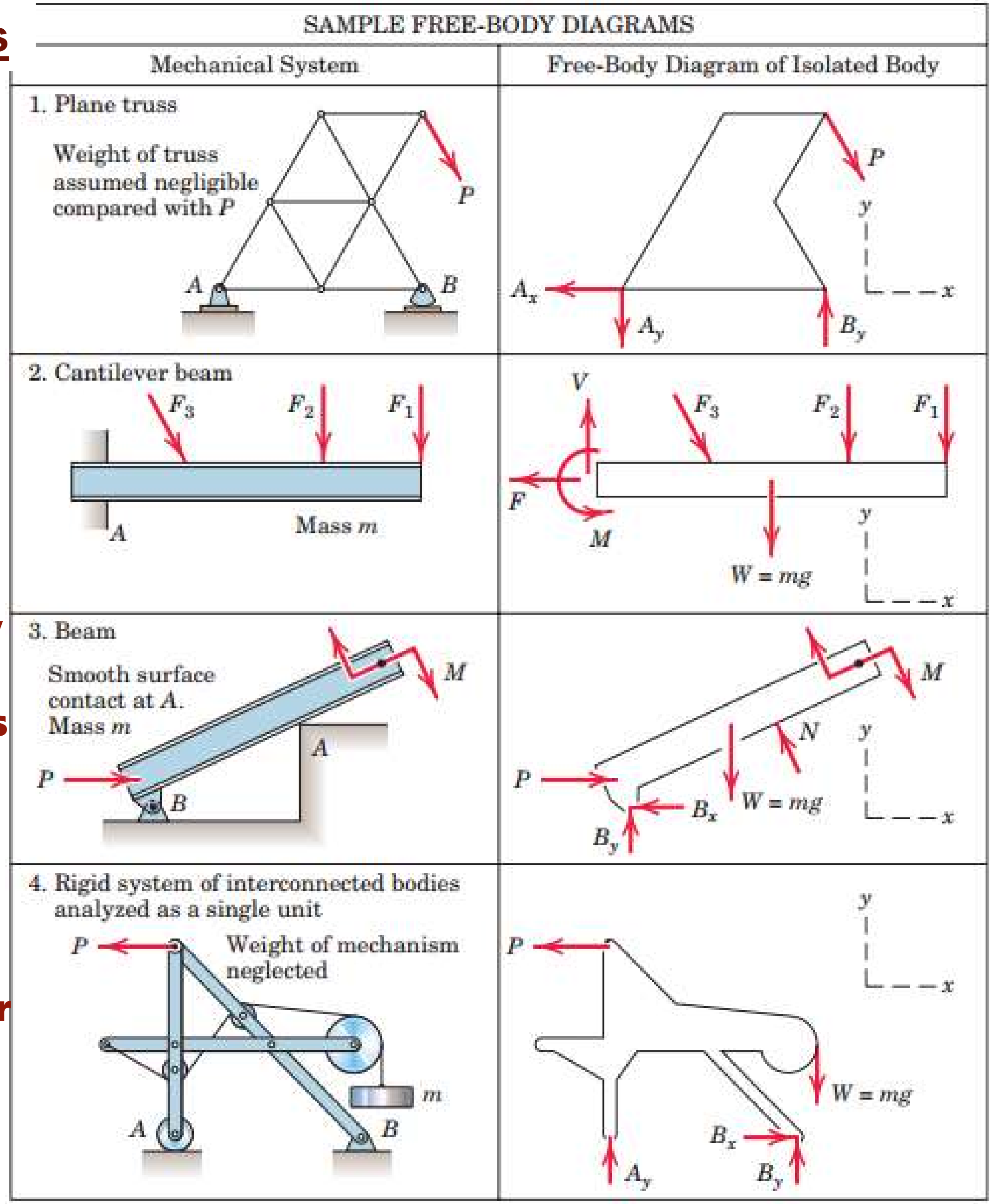
(*c*) Incomplete fixity
Partial constraints



(*d*) Excessive fixity
Redundant constraint

Construction of Free-Body Diagrams

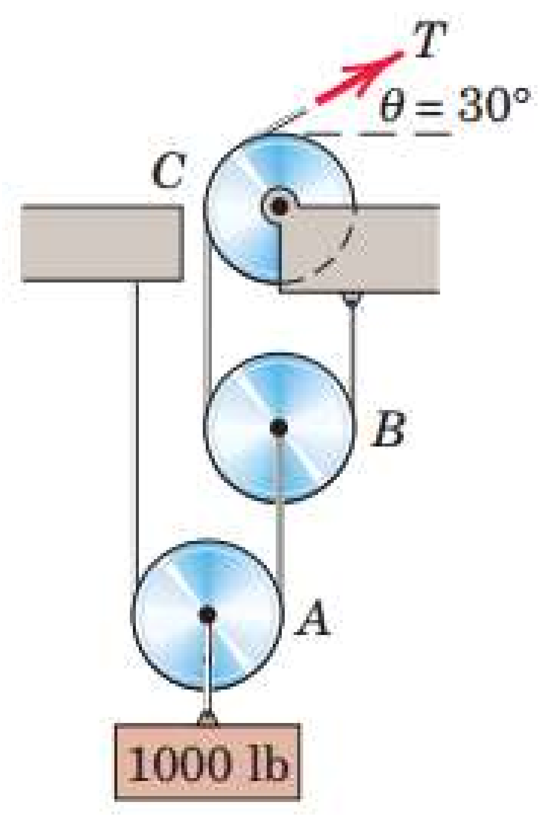
Step 1. Decide which system to isolate.
Step 2. Next isolate the chosen system by drawing a diagram which represents its *complete external boundary*.
Step 3. Identify all forces which act *on* the isolated system as applied *by* the removed contacting and attracting bodies, and represent them in their proper positions on the diagram of the isolated system. Make a systematic traverse of the entire boundary to identify all contact forces. Include body forces such as weights, where appreciable. Represent all known forces by vector arrows, each with its proper magnitude, direction, and sense indicated.
Step 4. Show the choice of coordinate axes directly on the diagram. Pertinent dimensions may also be represented for convenience.



SECTION A EQUILIBRIUM IN TWO DIMENSIONS

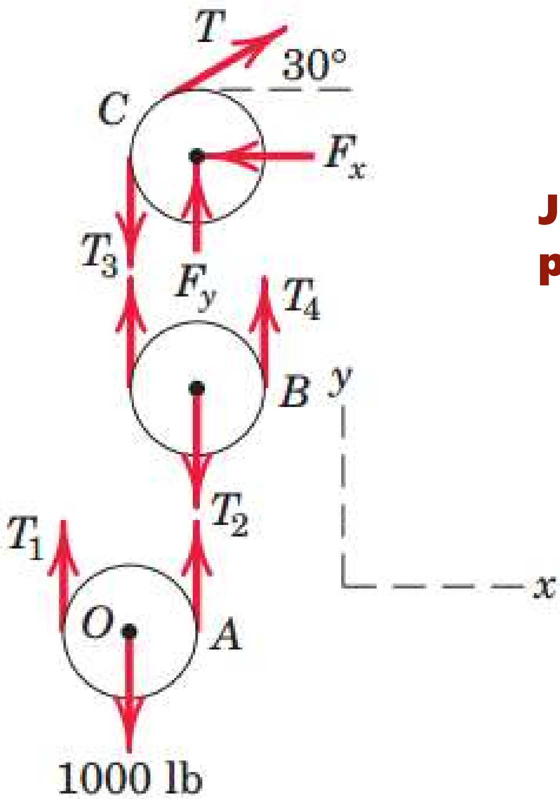
Example:

1) Calculate the tension T in the cable which supports the 1000-lb load with the pulley arrangement shown. Each pulley is free to rotate about its bearing, and the weights of all parts are small compared with the load. Find the magnitude of the total force on the bearing of pulley C.



SECTION A EQUILIBRIUM IN TWO DIMENSIONS

Solution. The free-body diagram of each pulley is drawn in its relative position to the others. We begin with pulley A, which includes the only known force. With the unspecified pulley radius designated by r, the equilibrium of moments about its center O and the equilibrium of forces in the vertical direction require



$$[\Sigma M_O = 0] \qquad T_1 r - T_2 r = 0 \quad T_1 = T_2$$

$$[\Sigma F_y = 0] \quad T_1 + T_2 - 1000 = 0 \quad 2T_1 = 1000 \quad T_1 = T_2 = 500 \text{ lb}$$

Just like pulley A we may write the equilibrium of forces on pulley B . Simply by inspection

$$T_3 = T_4 = T_2/2 = 250 \text{ lb}$$

For pulley C the angle ϴ= 30 in no way affects the moment of T about the center of the pulley, so that moment equilibrium requires

$$T = T_3 \qquad \text{or} \qquad T = 250 \text{ lb}$$

Equilibrium of the pulley in the x- and y-directions requires

$$[\Sigma F_x = 0] \qquad 250 \cos 30° - F_x = 0 \qquad F_x = 217 \text{ lb}$$

$$[\Sigma F_y = 0] \qquad F_y + 250 \sin 30° - 250 = 0 \qquad F_y = 125 \text{ lb}$$

$$\left[F = \sqrt{F_x^2 + F_y^2}\right] \quad F = \sqrt{(217)^2 + (125)^2} = 250 \text{ lb}$$

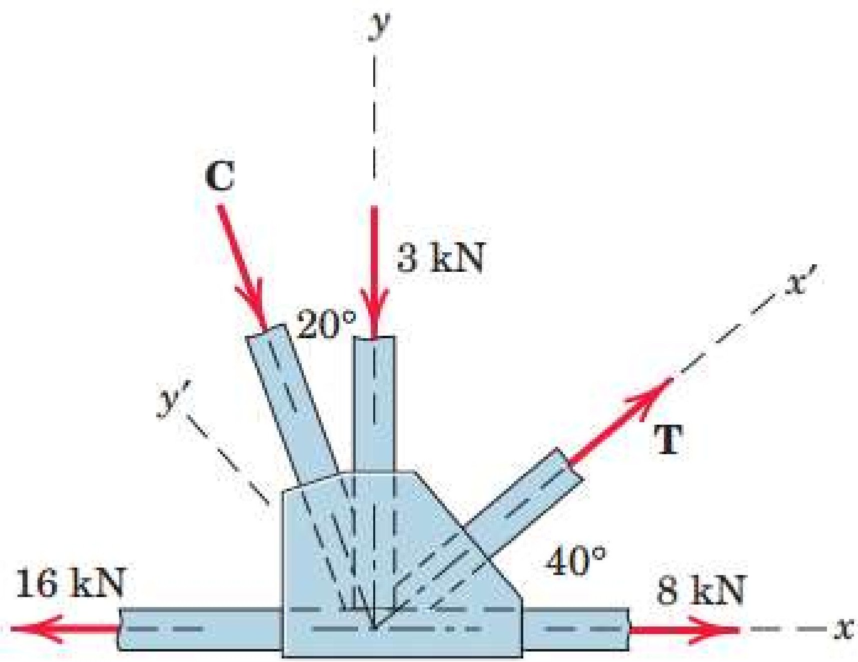
(Ans)

Mubarek Z

SECTION A EQUILIBRIUM IN TWO DIMENSIONS

Example:

2) Determine the magnitudes of the forces C and T, which, along with the other three forces shown, act on the bridge-truss joint.



Solution. The given sketch constitutes the free-body diagram of the isolated section of the joint in question and shows the five forces which are in equilibrium.

For the x-y axes as shown we have

$[\Sigma F_x = 0]$ $\qquad 8 + T\cos 40° + C\sin 20° - 16 = 0$

$$0.766T + 0.342C = 8$$

$[\Sigma F_y = 0]$ $\qquad T\sin 40° - C\cos 20° - 3 = 0$

$$0.643T - 0.940C = 3$$

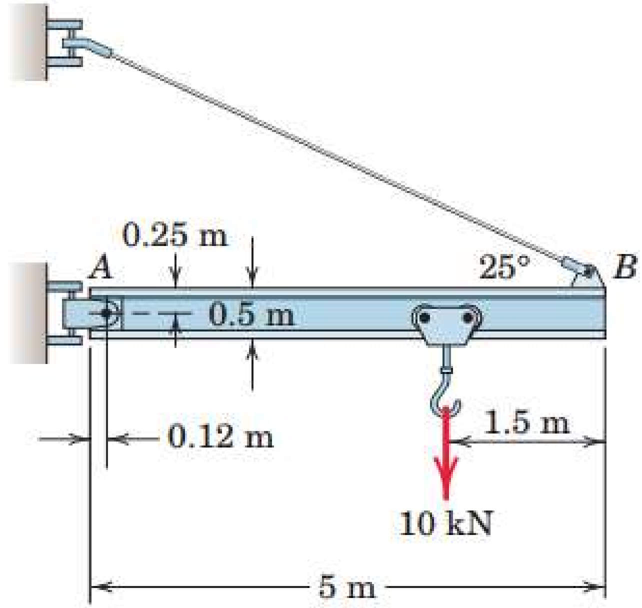
Simultaneous solution of Eqs. (a) and (b) produces

$$T = 9.09\ \text{kN} \qquad C = 3.03\ \text{kN}$$

SECTION A EQUILIBRIUM IN 2DS

Example:

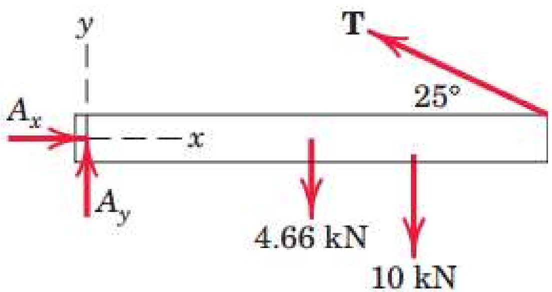
3) Determine the magnitude T of the tension in the supporting cable and the magnitude of the force on the pin at A for the jib crane shown. The beam AB is a standard 0.5-m I-beam with a mass of 95 kg per meter of length.



Solution:

The weight of the beam is $95(10^{-3})*5*9.81 = 4.66$ kN and acts through its center.

The free body diagram(FBD) is



Let`s take moment about A, the counterclockwise sense as positive we write $[\Sigma M_A = 0]$

$$(T \cos 25°)0.25 + (T \sin 25°)(5 - 0.12) - 10(5 - 1.5 - 0.12) - 4.66(2.5 - 0.12) = 0$$

$$T = 19.61 \text{ kN}$$

Equating the sums of forces in the x- and y-directions to zero gives

$$[\Sigma F_x = 0] \qquad A_x - 19.61 \cos 25° = 0 \qquad A_x = 17.77 \text{ kN}$$

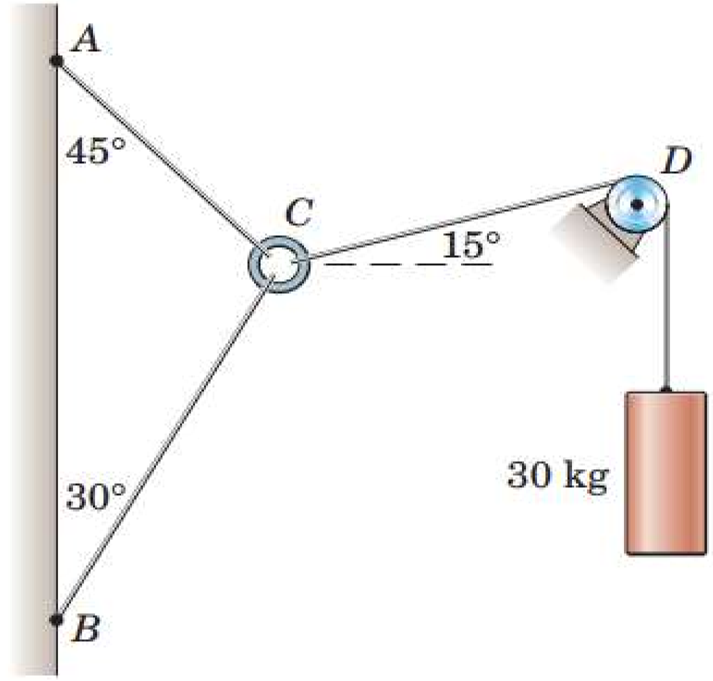
$$[\Sigma F_y = 0] \qquad A_y + 19.61 \sin 25° - 4.66 - 10 = 0 \qquad A_y = 6.37 \text{ kN}$$

$$\left[A = \sqrt{A_x^2 + A_y^2}\right] \qquad A = \sqrt{(17.77)^2 + (6.37)^2} = 18.88 \text{ kN}$$

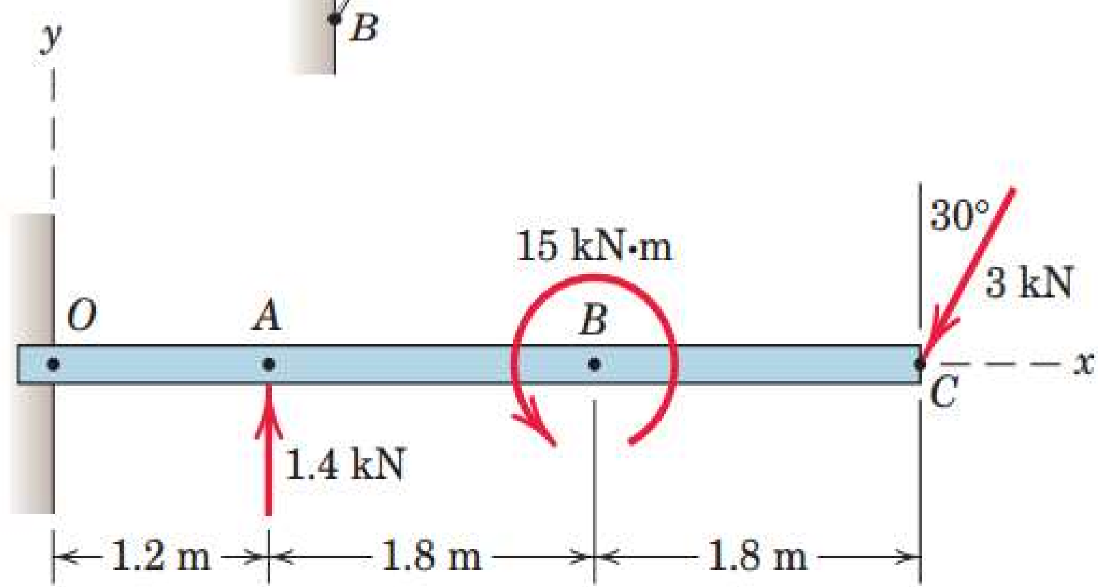
SECTION A EQUILIBRIUM IN TWO 2DS…

EXERCISE

1) Three cables are joined at the junction ring C. Determine the tensions in cables AC and BC caused by the weight of the 30-kg cylinder.



2)The 500-kg uniform beam is subjected to the three external loads shown. Compute the reactions at the support point O. The x-y plane is vertical.



SECTION B EQUILIBRIUM IN THREE DIMENSIONS

Equilibrium Conditions

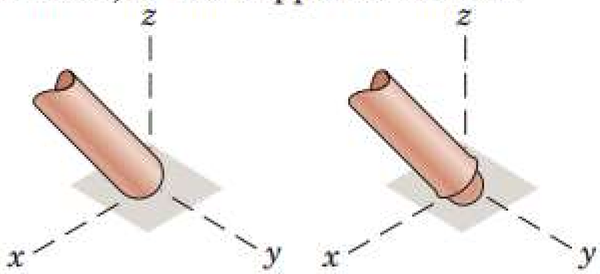
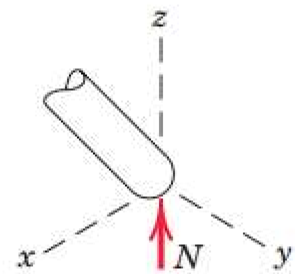
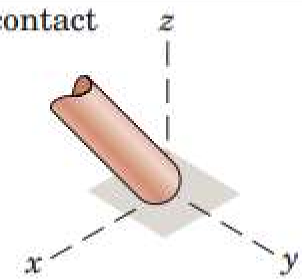
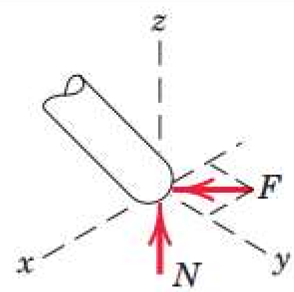
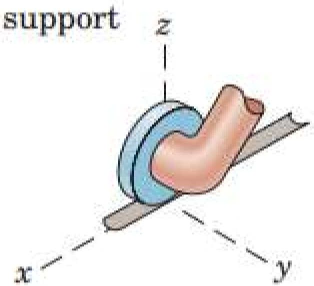
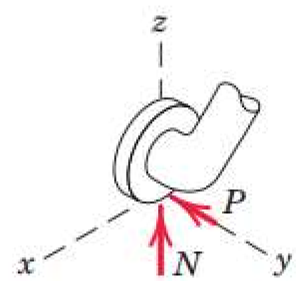
- ✓ **The general conditions for the equilibrium of a body require that the resultant force and resultant couple on a body in equilibrium be zero. These two vector equations of equilibrium and their scalar components may be written as**

$$\Sigma \mathbf{F} = \mathbf{0} \qquad \text{or} \qquad \begin{cases} \Sigma F_x = 0 \\ \Sigma F_y = 0 \\ \Sigma F_z = 0 \end{cases}$$

$$\Sigma \mathbf{M} = \mathbf{0} \qquad \text{or} \qquad \begin{cases} \Sigma M_x = 0 \\ \Sigma M_y = 0 \\ \Sigma M_z = 0 \end{cases}$$

These six equations are both necessary and sufficient conditions for complete equilibrium. The reference axes may be chosen arbitrarily as a matter of convenience, the only restriction being that a right-handed coordinate system should be chosen when vector notation is used.

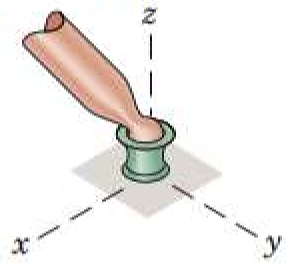
SECTION B EQUILIBRIUM IN 3Ds

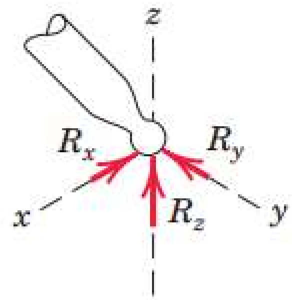
MODELING THE ACTION OF FORCES IN THREE-DIMENSIONAL ANALYSIS		
Type of Contact and Force Origin	**Action on Body to Be Isolated**	
1. Member in contact with smooth surface, or ball-supported member 		Force must be normal to the surface and directed toward the member.
2. Member in contact with rough surface 		The possibility exists for a force F tangent to the surface (friction force) to act on the member, as well as a normal force N.
3. Roller or wheel support with lateral constraint 		A lateral force P exerted by the guide on the wheel can exist, in addition to the normal force N.

3. EQUILIBRIUM…

SECTION B EQUILIBRIUM IN 3DS…

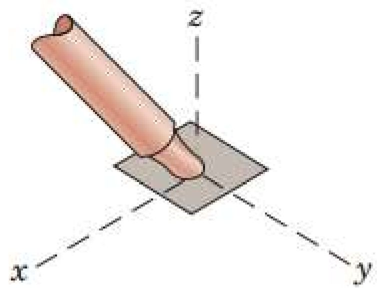
4. Ball-and-socket joint

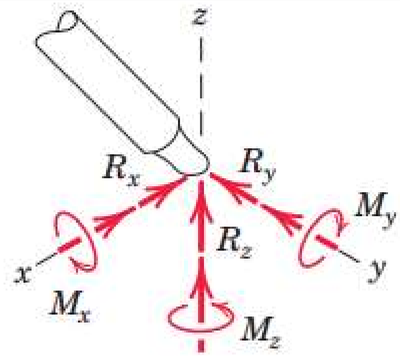




A ball-and-socket joint free to pivot about the center of the ball can support a force **R** with all three components.

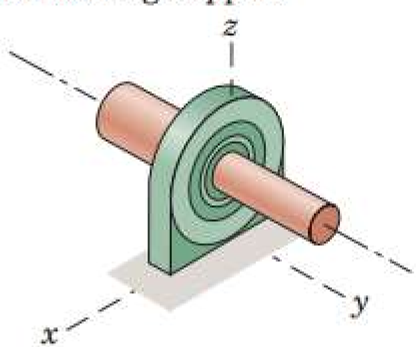
5. Fixed connection (embedded or welded)

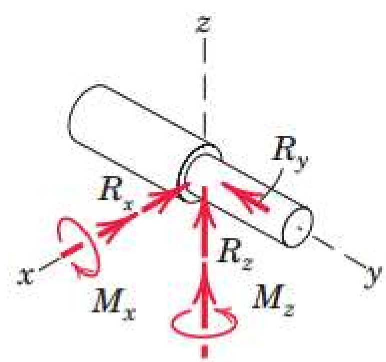




In addition to three components of force, a fixed connection can support a couple **M** represented by its three components.

6. Thrust-bearing support

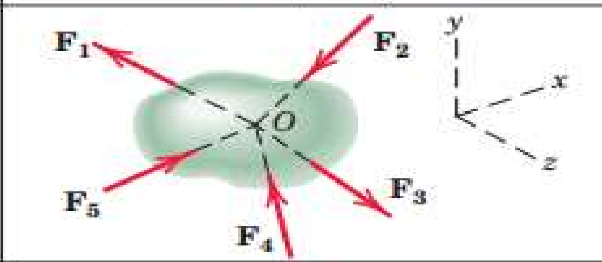
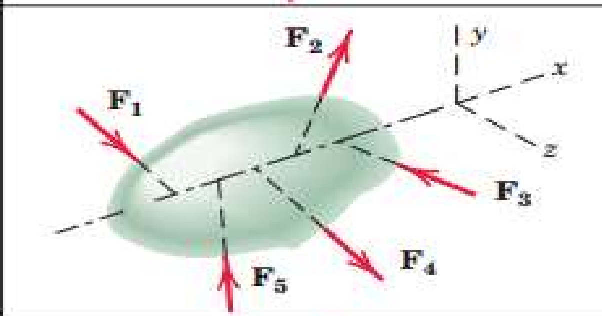
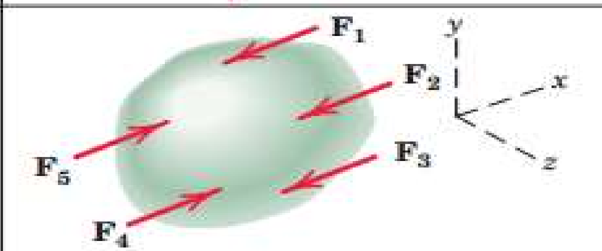
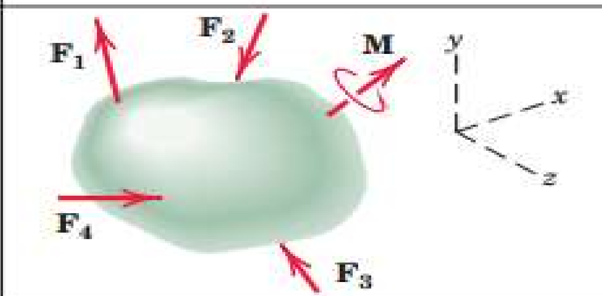




Thrust bearing is capable of supporting axial force R_y as well as radial forces R_x and R_z. Couples M_x and M_z must, in some cases, be assumed zero in order to provide statical determinacy.

3. EQUILIBRIUM…

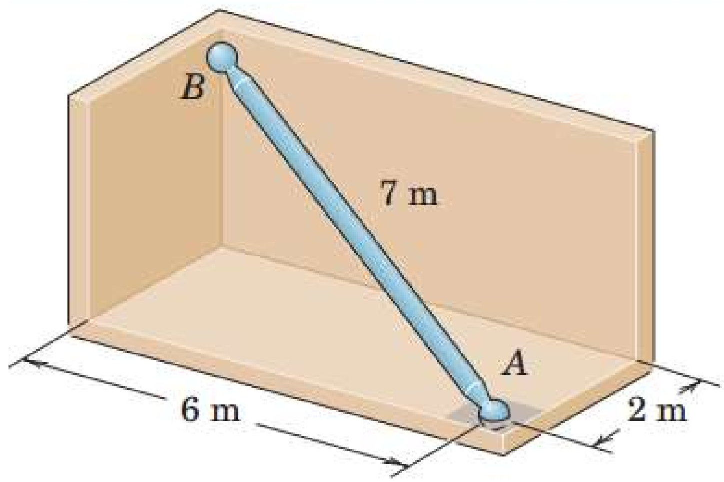
SECTION B EQUILIBRIUM IN 3Ds…

CATEGORIES OF EQUILIBRIUM IN THREE DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$ $\Sigma F_z = 0$
2. Concurrent with a line		$\Sigma F_x = 0$ $\Sigma M_y = 0$ $\Sigma F_y = 0$ $\Sigma M_z = 0$ $\Sigma F_z = 0$
3. Parallel		$\Sigma F_x = 0$ $\Sigma M_y = 0$ $\Sigma M_z = 0$
4. General		$\Sigma F_x = 0$ $\Sigma M_x = 0$ $\Sigma F_y = 0$ $\Sigma M_y = 0$ $\Sigma F_z = 0$ $\Sigma M_z = 0$

SECTION B EQUILIBRIUM IN 3Ds…

Example

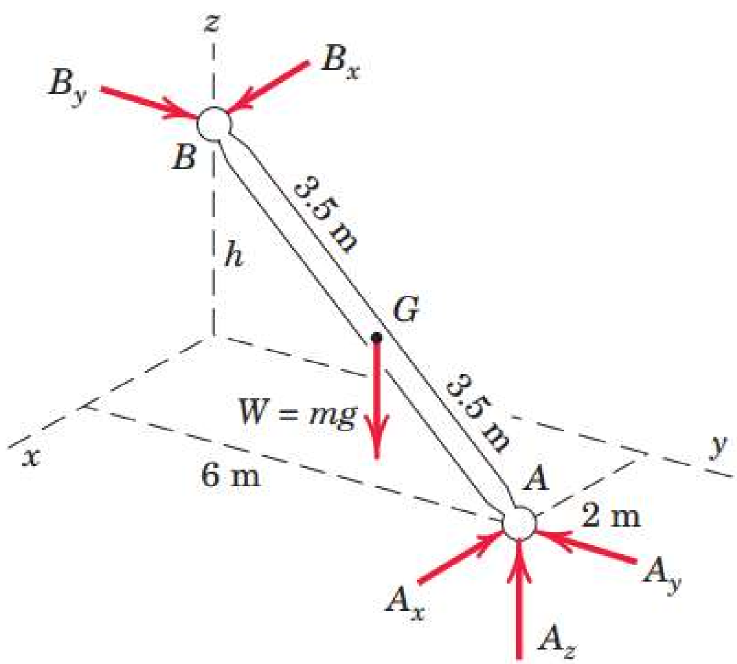
The uniform 7-m steel shaft has a mass of 200 kg and is supported by a ball and-socket joint at A in the horizontal floor. The ball end B rests against the smooth vertical walls as shown. Compute the forces exerted by the walls and the floor on the ends of the shaft.



Solution:

$W = mg = 200(9.81) = 1962$ N

The free body diagram is as shown below



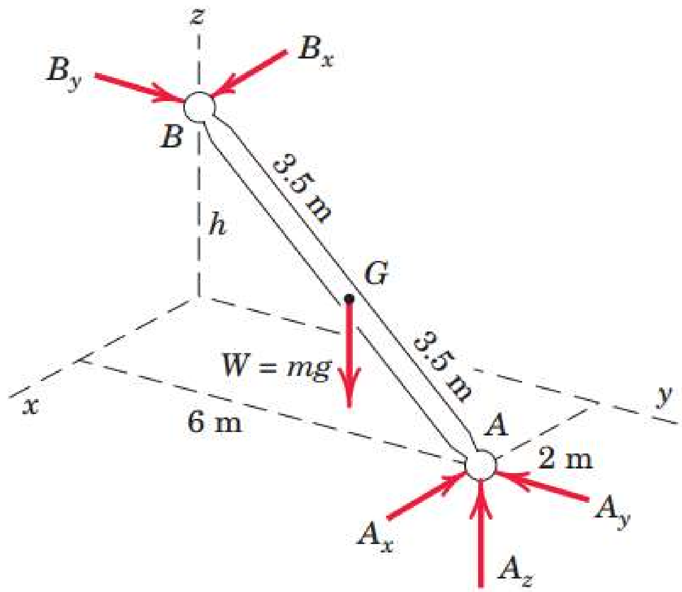
The vertical position of B is found from

$$7 = \sqrt{2^2 + 6^2 + h^2}, \qquad h=3\text{m}$$

SECTION B EQUILIBRIUM IN 3Ds…

Solution: ***continued …***

Scalar solution: Evaluating the scalar moment equations about axes through A parallel, respectively, to the x- and y-axes, gives



$[\Sigma M_{A_x} = 0] \qquad 1962(3) - 3B_y = 0 \qquad B_y = 1962 \text{ N}$

$[\Sigma M_{A_y} = 0] \qquad -1962(1) + 3B_x = 0 \qquad B_x = 654 \text{ N}$

The force equations give, simply,

$[\Sigma F_x = 0] \qquad -A_x + 654 = 0 \qquad A_x = 654 \text{ N}$

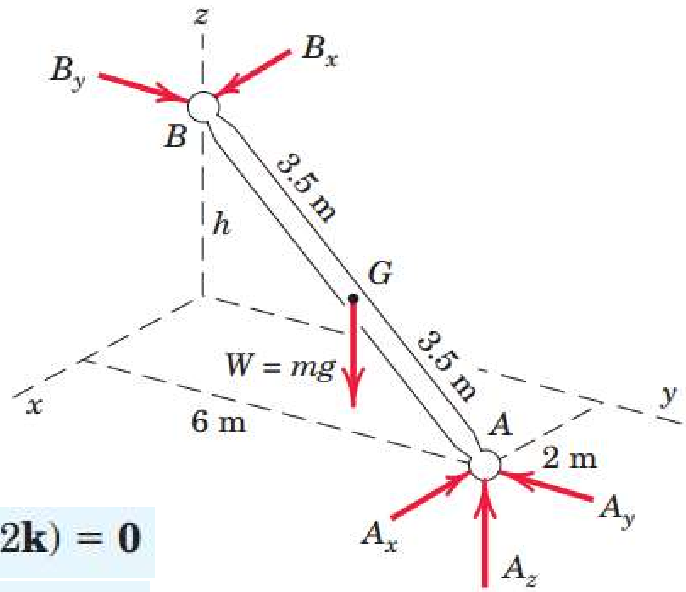
$[\Sigma F_y = 0] \qquad -A_y + 1962 = 0 \qquad A_y = 1962 \text{ N}$

$[\Sigma F_z = 0] \qquad A_z - 1962 = 0 \qquad A_z = 1962 \text{ N}$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$
$$= \sqrt{(654)^2 + (1962)^2 + (1962)^2} = 2850 \text{ N}$$

SECTION B EQUILIBRIUM IN 3Ds…

Solution: *continued …*



Vector solution: We will use A as a moment center to eliminate reference to the forces at A.

- ***The position vectors needed to compute the moments about A are***

$$\mathbf{r}_{AG} = -1\mathbf{i} - 3\mathbf{j} + 1.5\mathbf{k} \text{ m} \qquad \mathbf{r}_{AB} = -2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k} \text{ m}$$

$$[\Sigma \mathbf{M}_A = \mathbf{0}] \qquad \mathbf{r}_{AB} \times (\mathbf{B}_x + \mathbf{B}_y) + \mathbf{r}_{AG} \times \mathbf{W} = \mathbf{0}$$

$$(-2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j}) + (-\mathbf{i} - 3\mathbf{j} + 1.5\mathbf{k}) \times (-1962\mathbf{k}) = \mathbf{0}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -6 & 3 \\ B_x & B_y & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -3 & 1.5 \\ 0 & 0 & -1962 \end{vmatrix} = \mathbf{0}$$

$$(-3B_y + 5890)\mathbf{i} + (3B_x - 1962)\mathbf{j} + (-2B_y + 6B_x)\mathbf{k} = \mathbf{0}$$

Equating the coefficients of i, j, and k to zero and solving give $B_x = 654 \text{ N}$ $\quad B_y = 1962 \text{ N}$

✓ **The forces at A are easily determined by**

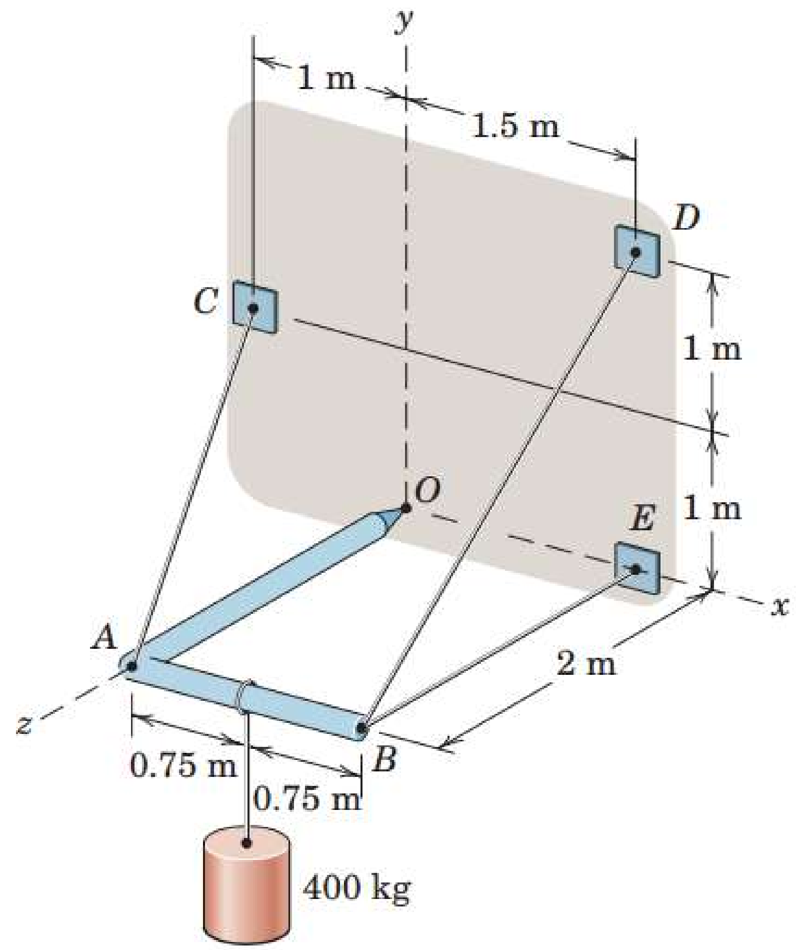
$$[\Sigma \mathbf{F} = \mathbf{0}] \qquad (654 - A_x)\mathbf{i} + (1962 - A_y)\mathbf{j} + (-1962 + A_z)\mathbf{k} = \mathbf{0}$$

and $A_x = 654 \text{ N} \qquad A_y = 1962 \text{ N} \qquad A_z = 1962 \text{ N}$

SECTION B EQUILIBRIUM IN 3Ds…

Exercise:

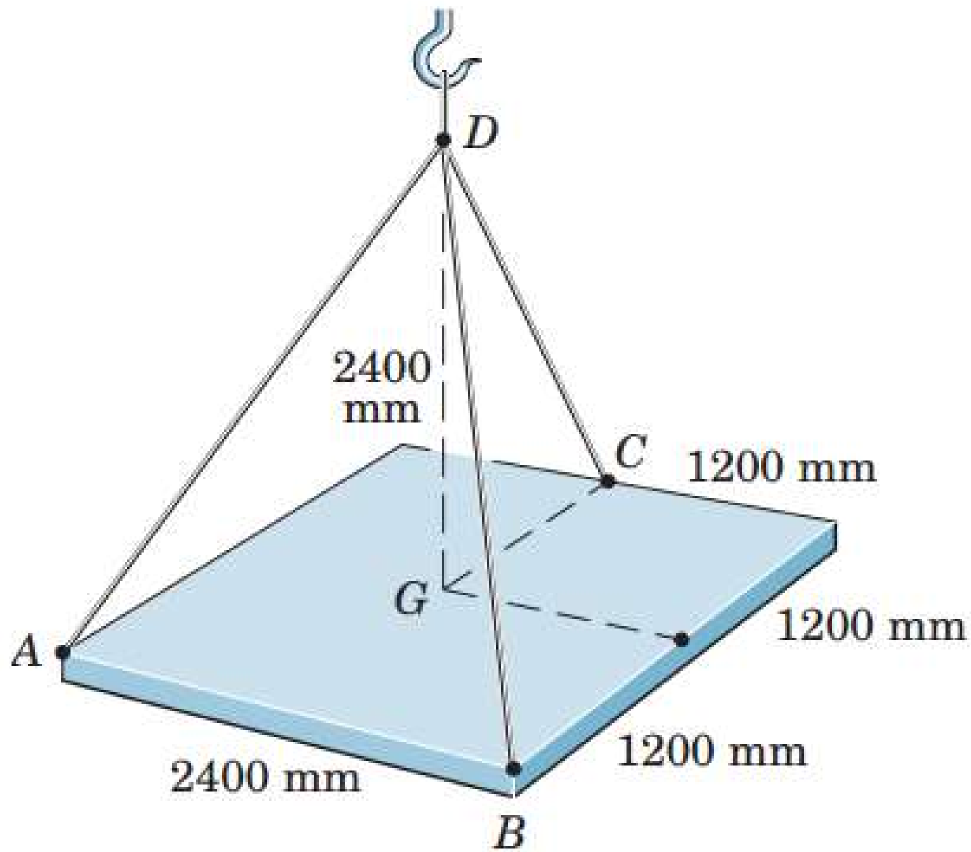
1) The light right-angle boom which supports the 400-kg cylinder is supported by three cables and a ball and-socket joint at O attached to the vertical x-y surface. Determine the reactions at O and the cable tensions



SECTION B EQUILIBRIUM IN 3Ds…

Exercise:

3) The square steel plate has a mass of 1800 kg with mass center at its center G. Calculate the tension in each of the three cables with which the plate is lifted while remaining horizontal.



Introduction

In Chapter 3 above, we studied the equilibrium of a single rigid body or a system of connected members treated as a single rigid body. We first drew a free-body diagram of the body showing all forces external to the isolated body and then we applied the force and moment equations of equilibrium to determine unknown external reactions.

In this chaper, we focus on the determination of the forces internal to a structure—that is, forces of action and reaction between the connected members.

An engineering structure is any connected system of members built to support or transfer forces and to safely withstand the loads applied to it. To determine the forces internal to an engineering structure, we must dismember the structure and analyze separate free-body diagrams of individual members or combinations of members.

This analysis requires careful application of Newton's third law, which states that each action is accompanied by an equal and opposite reaction.

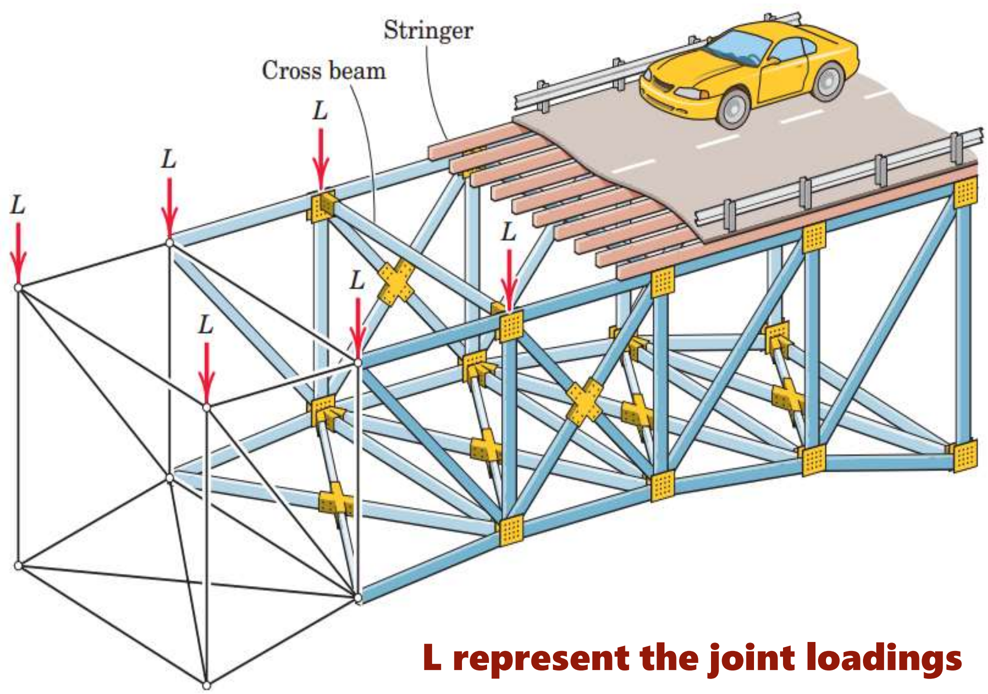
we analyze the internal forces acting in several types of structures—namely, trusses, frames, and machines. In this treatment

we consider only statically determinate structures, which do not have more supporting constraints than are necessary to maintain an equilibrium configuration. Thus, as we have already seen, the equations of equilibrium are adequate to determine all unknown reactions.

Plane Trusses

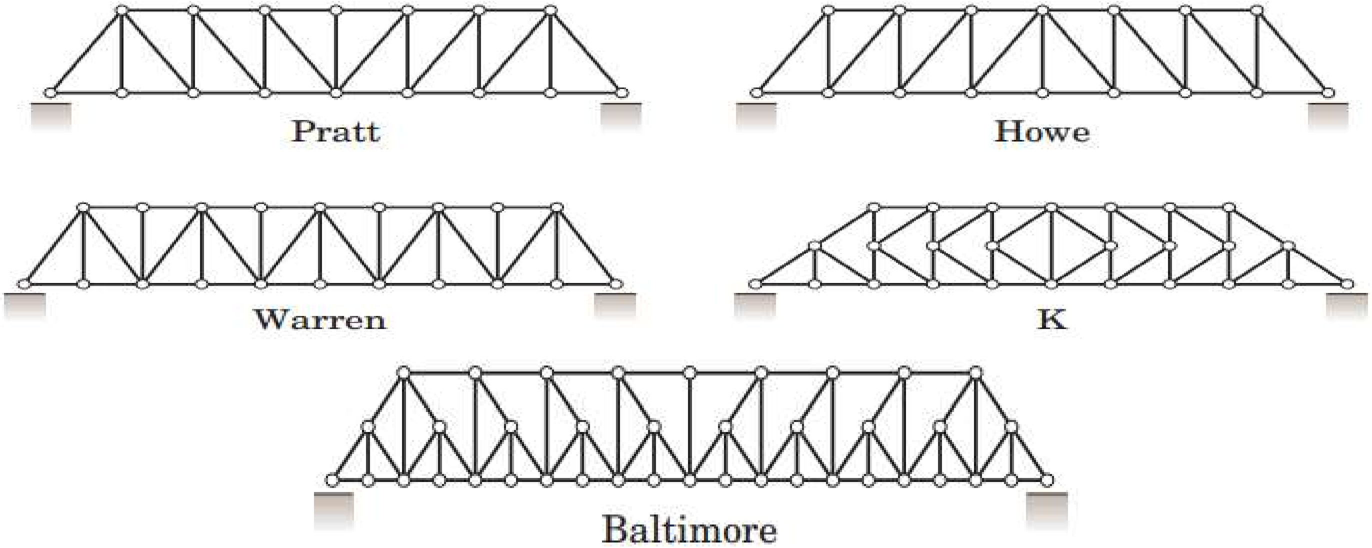
A framework composed of members joined at their ends to form a rigid structure is called a truss.
Bridges, roof supports, derricks, and other such structures are common examples of trusses.
Structural members commonly used are I-beams, channels, angles, bars, and special shapes which are fastened together at their ends by welding, riveted connections, or large bolts or pins. When the members of the truss lie essentially in a single plane, the truss is called a plane truss.

For bridges and similar structures, plane trusses are commonly utilized in pairs with one truss assembly placed on each side of the structure. The combined weight of the roadway and vehicles is transferred to the longitudinal stringers, then to the cross beams, and finally, with the weights of the stringers and cross beams accounted for, to the upper joints of the two plane trusses which form the vertical sides of the structure.

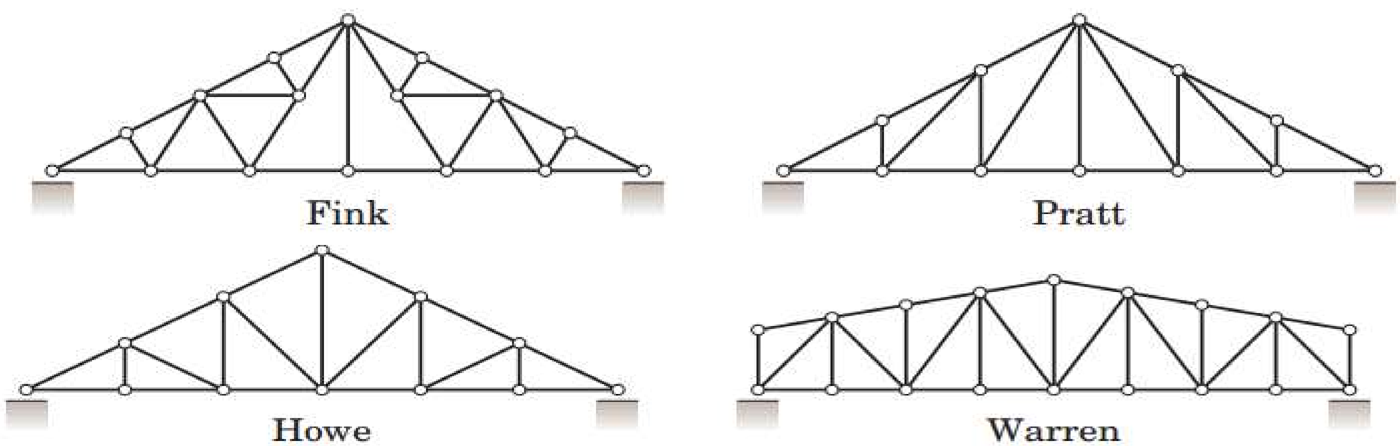


L represent the joint loadings

Plane Trusses



Commonly Used Bridge Trusses



Commonly Used Roof Trusses

Plane Trusses…

Simple Truss

The basic element of a plane truss is the triangle. Three bars joined by pins at their ends, Fig. 4/3a, constitute a rigid frame. The term rigid is used to mean non-collapsible and also to mean that deformation of the members due to induced internal strains is negligible. On the other hand, four or more bars pin-jointed to form a polygon of as many sides constitute a nonrigid frame. We can make the nonrigid frame in Fig. 4/3b rigid, or stable, by adding a diagonal bar joining A and D or B and C and thereby forming two triangles. We can extend the structure by adding additional units of two end-connected bars, such as DE and CE or AF and DF, Fig. 4/3c, which are pinned to two fixed joints. In this way the entire structure will remain rigid. Structures built from a basic triangle in the manner described are known as simple trusses. When more members are present than are needed to prevent collapse, the truss is statically indeterminate. A statically indeterminate truss cannot be analyzed by the equations of equilibrium alone. Additional members or supports which are not necessary for maintaining the equilibrium configuration are called redundant.

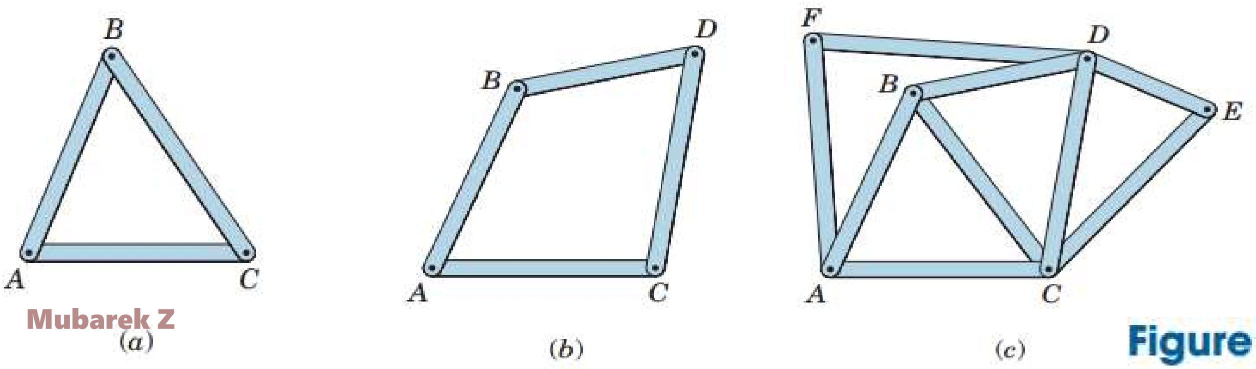
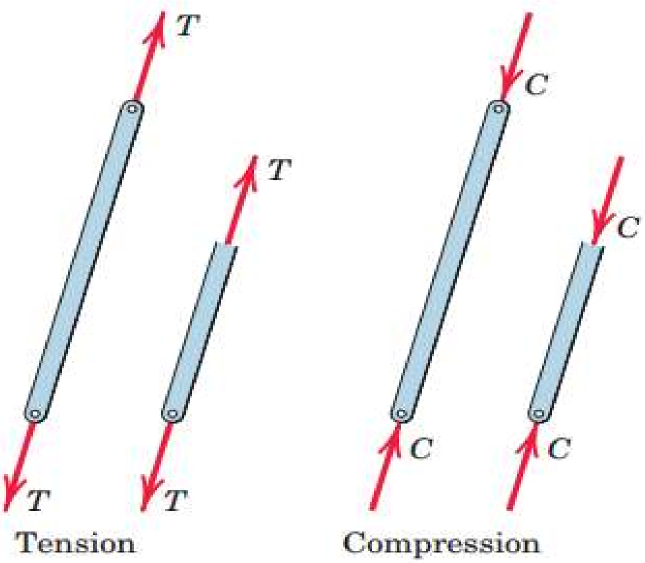


Figure 4/3

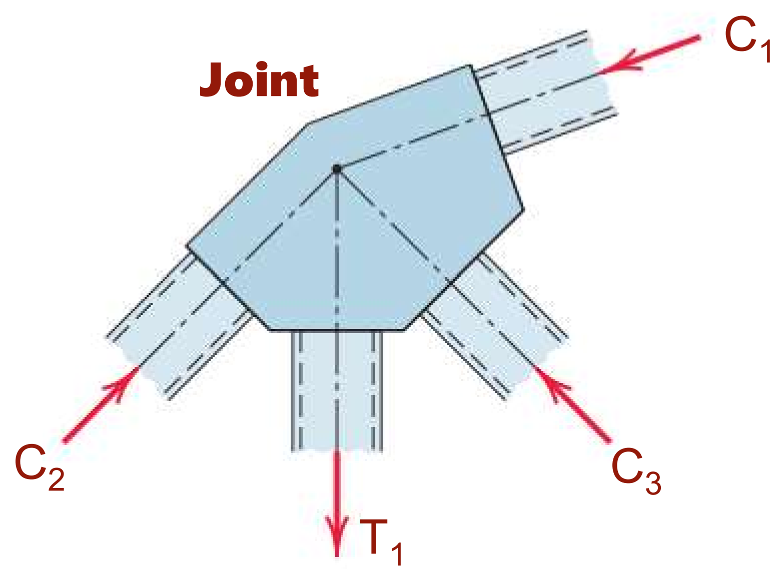
Plane Trusses…

Simple Truss

To design a truss we must first determine the forces in the various members and then select appropriate sizes and structural shapes to withstand the forces. Several assumptions are made in the force analysis of simple trusses. First, we assume all members to be two-force members. A two-force member is one in equilibrium under the action of two forces only. Each member of a truss is normally a straight link joining the two points of application of force. The two forces are applied at the ends of the member and are necessarily equal, opposite, and collinear for equilibrium. When we represent the equilibrium of a portion of a two-force member, the tension T or compression C acting on the cut section is the same for all sections.

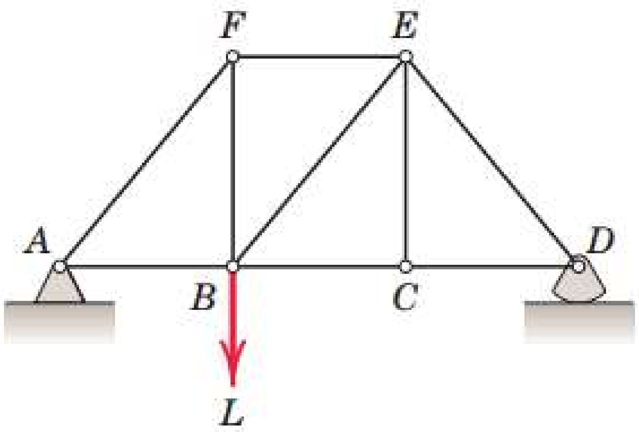


Two-Force Members

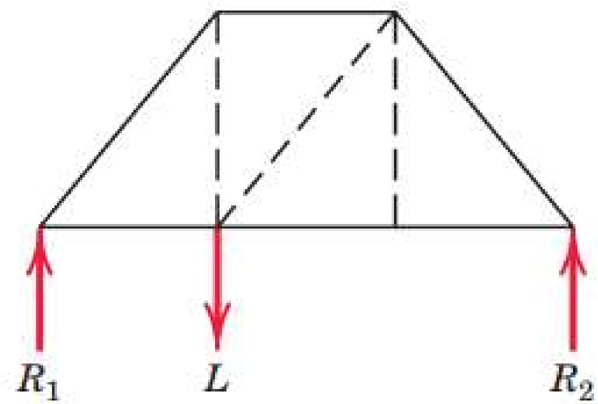


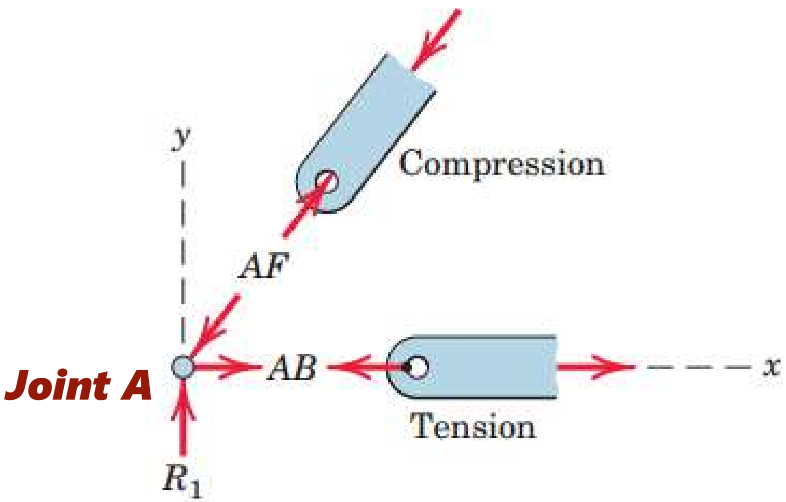
Plane Trusses…

Method of Joints



Frist, analyze the whole system externally for determining external reactions.





Second, analyze internally by taking joint forces We begin the analysis with any joint where at least one known load exists and where not more than two unknown forces are present. The solution may be started with the pin at the left end.

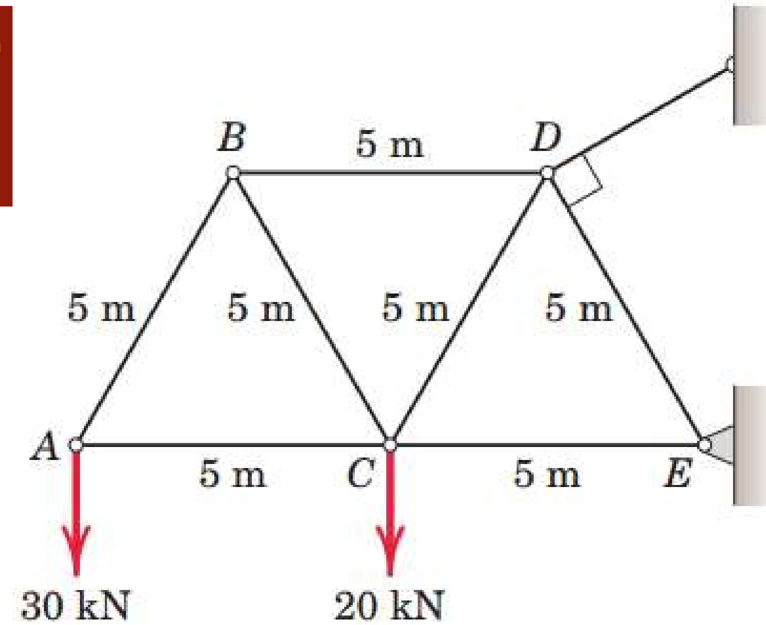
To determine member force AF, & AB

$$\Sigma F_x = 0 \quad \Sigma F_y = 0$$

Similarly Joint F is to be analyzed and then joint B follows. Finally joint E and C can be analyzed.

Plane Trusses…

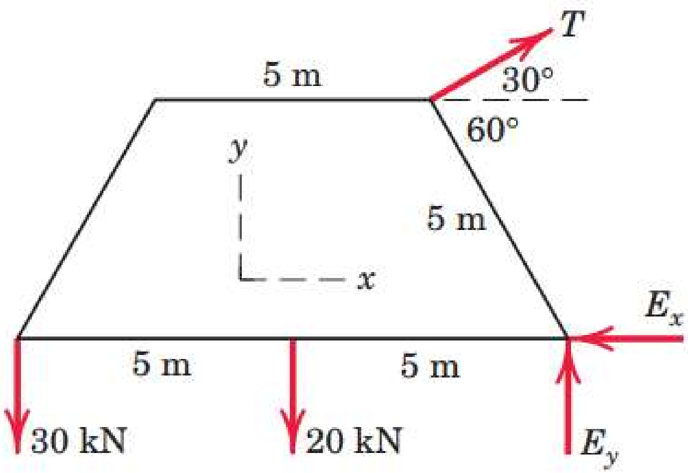
Example: **Compute the force in each member of the loaded cantilever truss by the method of joints**



Solution:

Steps 1: Analyze the whole system externally i.e determine external reactions at D & E

- **By using the whole free body diagram**



The equations of equilibrium give

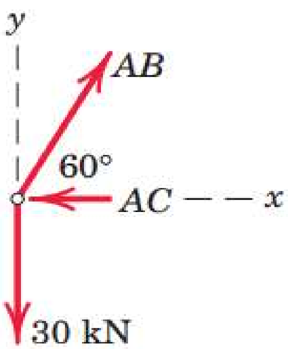
$[\Sigma M_E = 0]$	$5T - 20(5) - 30(10) = 0$	$T = 80$ kN
$[\Sigma F_x = 0]$	$80 \cos 30° - E_x = 0$	$E_x = 69.3$ kN
$[\Sigma F_y = 0]$	$80 \sin 30° + E_y - 20 - 30 = 0$	$E_y = 10$ kN

Plane Trusses…

Solution *continued …*

Steps 2: Analyze the truss joints internally i.e determine internal force in each member

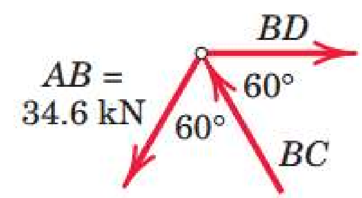
- ✓ **draw free-body diagrams showing the forces acting on each of the connecting pins (joints). Write equilibrium equations in x and y directions at each joints. Begin with the simple joint and continue consecutively by giving priority for simplicity of the joint.**



Joint A

$[\Sigma F_y = 0]$ $\quad 0.866AB - 30 = 0 \quad AB = 34.6 \text{ kN } T$

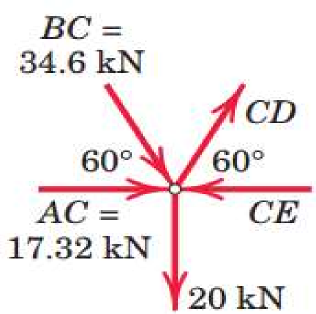
$[\Sigma F_x = 0]$ $\quad AC - 0.5(34.6) = 0 \quad AC = 17.32 \text{ kN } C$



Joint B

$[\Sigma F_y = 0]$ $\quad 0.866BC - 0.866(34.6) = 0 \quad BC = 34.6 \text{ kN } C$

$[\Sigma F_x = 0]$ $\quad BD - 2(0.5)(34.6) = 0 \quad BD = 34.6 \text{ kN } T$

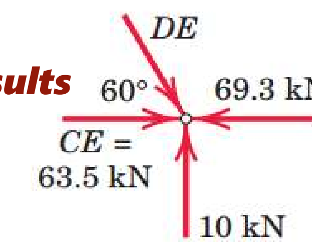


Joint C

$[\Sigma F_y = 0]$ $\quad CD = 57.7 \text{ kN } T$

$[\Sigma F_x = 0]$ $\quad CE = 63.5 \text{ kN } C$

Finally, from joint E there results



Joint E

$[\Sigma F_y = 0]$ $\quad 0.866DE = 10 \quad DE = 11.55 \text{ kN } C$

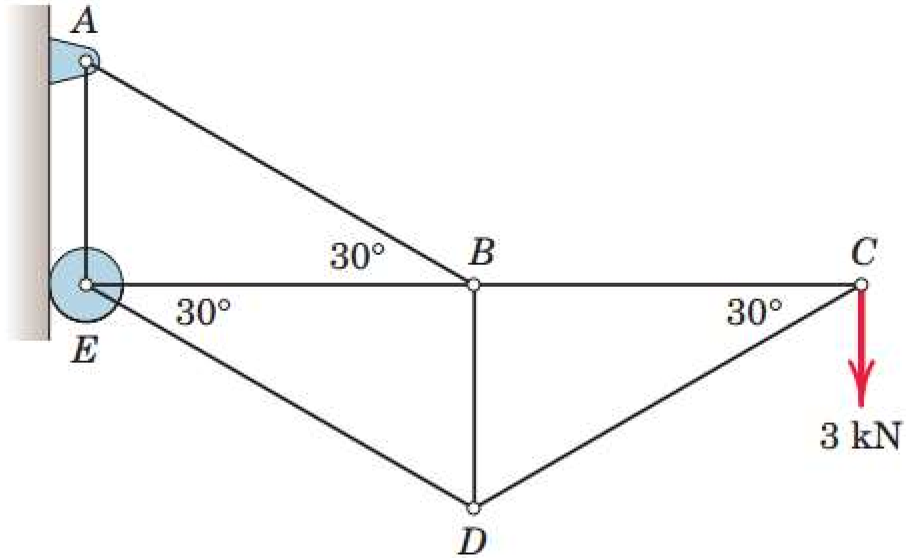
and the equation ΣFx = 0 checks

Plane Trusses…

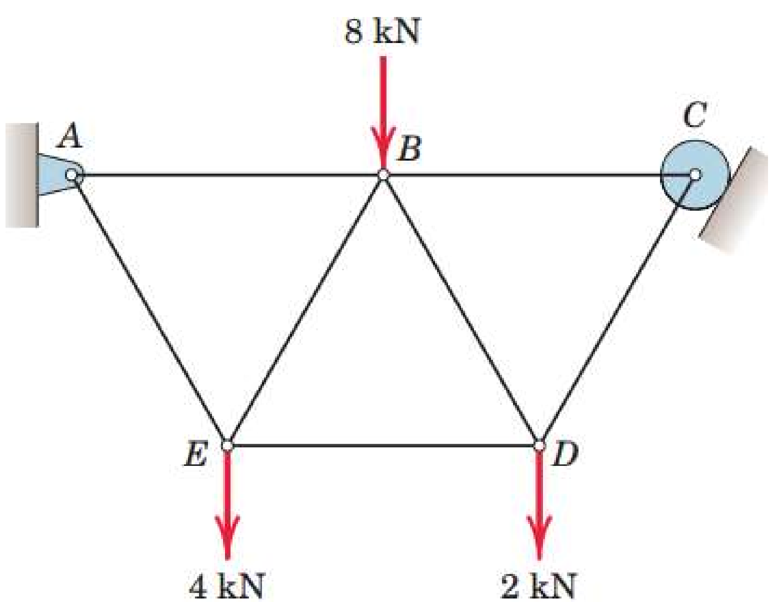
Method of joints

Exercise

1. Determine the force in each member of the loaded truss.



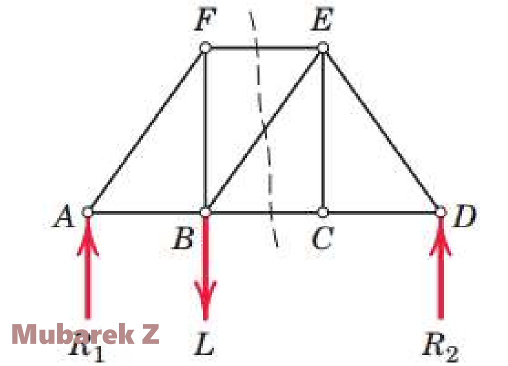
2. Determine the force in each member of the loaded truss. All triangles are equilateral.

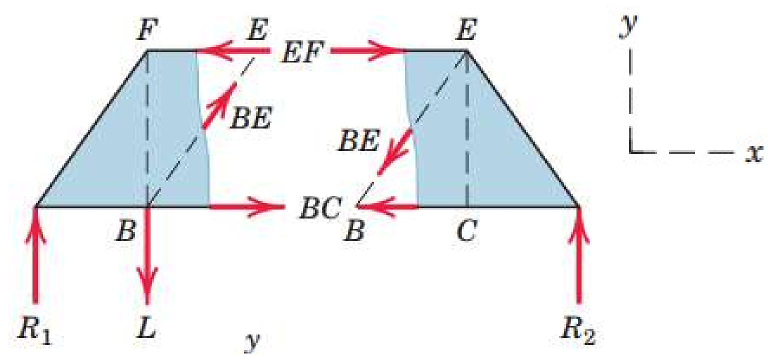


Plane Trusses…

Method of Section

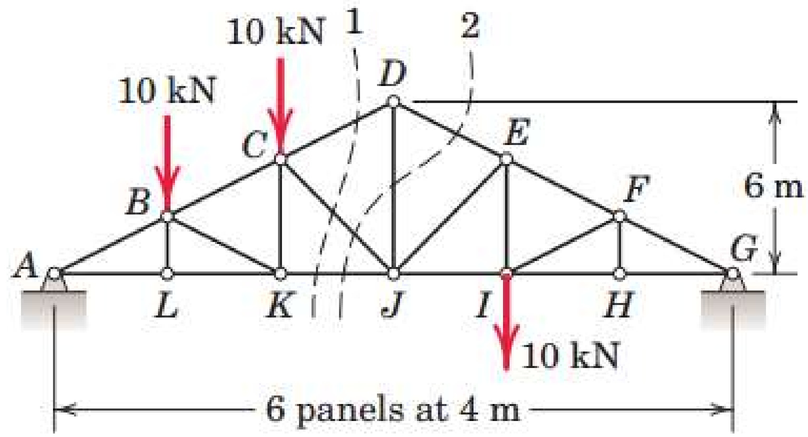
When analyzing plane trusses by the method of joints, we need only two of the three equilibrium equations because the procedures involve concurrent forces at each joint. We can take advantage of the third or moment equation of equilibrium by selecting an entire section of the truss for the free body in equilibrium under the action of a nonconcurrent system of forces. This method of sections has the basic advantage that the force in almost any desired member may be found directly from an analysis of a section which has cut that member. Thus, it is not necessary to proceed with the calculation from joint to joint until the member in question has been reached. In choosing a section of the truss, we note that, in general, not more than three members whose forces are unknown should be cut, since there are only three available independent equilibrium relations.



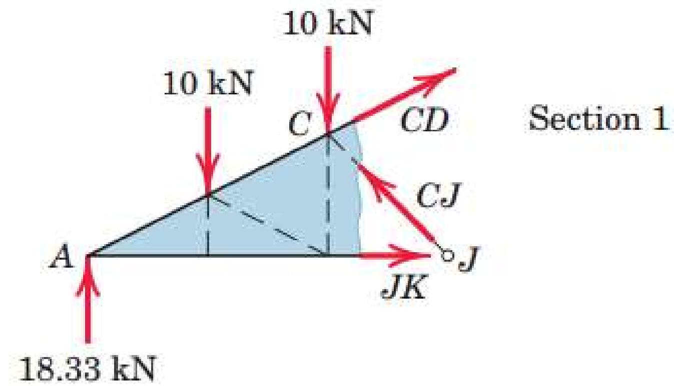


Plane Trusses…

Example: ***Calculate the force in member DJ of the Howe roof truss illustrated. Neglect any horizontal components of force at the supports.***



Solution: ***1st determine reactions then Use FBD of the considered sections***



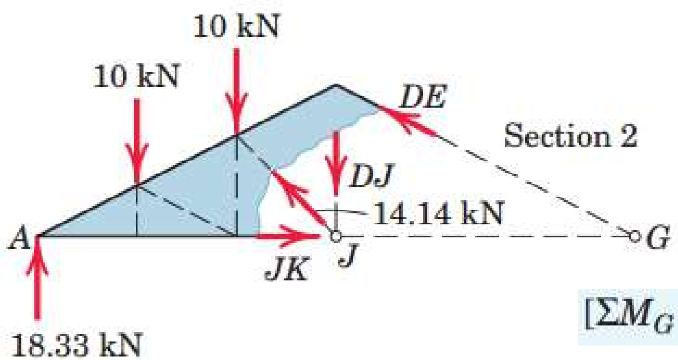
By the analysis of section 1, CJ is obtained from

$[\Sigma M_A = 0]$ $\quad 0.707CJ(12) - 10(4) - 10(8) = 0$

$CJ = 14.14 \text{ kN } C$

$[\Sigma M_J = 0]$ $\quad 0.894CD(6) + 18.33(12) - 10(4) - 10(8) = 0$

$CD = -18.63 \text{ kN}$ $\quad CD = 18.63 \text{ kN } C$



From the FBD of section 2, which now includes the known value of CJ, a balance of moments about G is seen to eliminate DE and JK. Thus,

$[\Sigma M_G = 0]$ $\quad 12DJ + 10(16) + 10(20) - 18.33(24) - 14.14(0.707)(12) = 0$

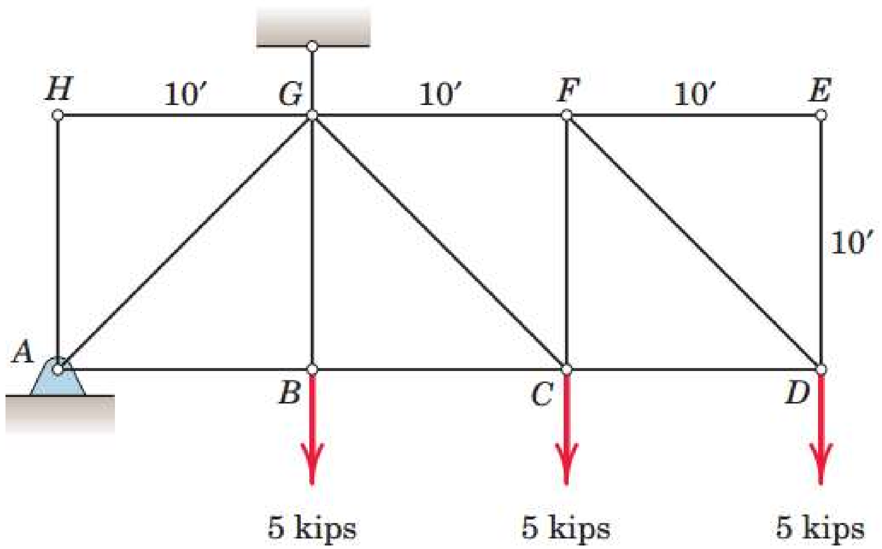
$DJ = 16.67 \text{ kN } T$ *Ans.*

Mubarek Z

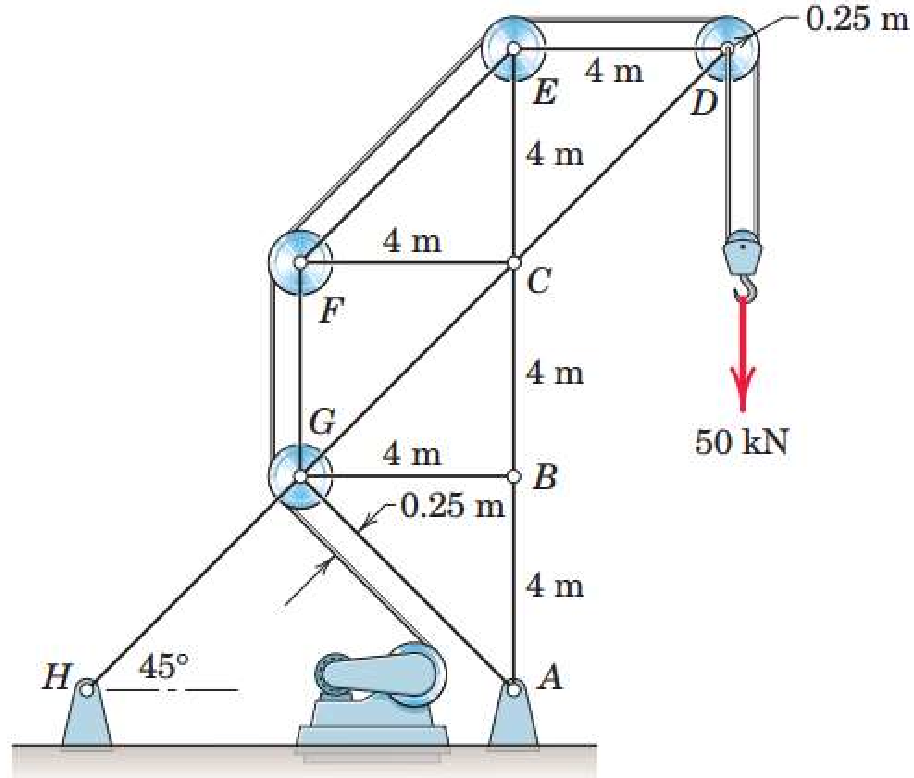
Plane Trusses…

Exercise

1. ***Determine the force in member CG***



2. Determine the forces in members FG, CG, BC, and EF for the loaded crane truss.



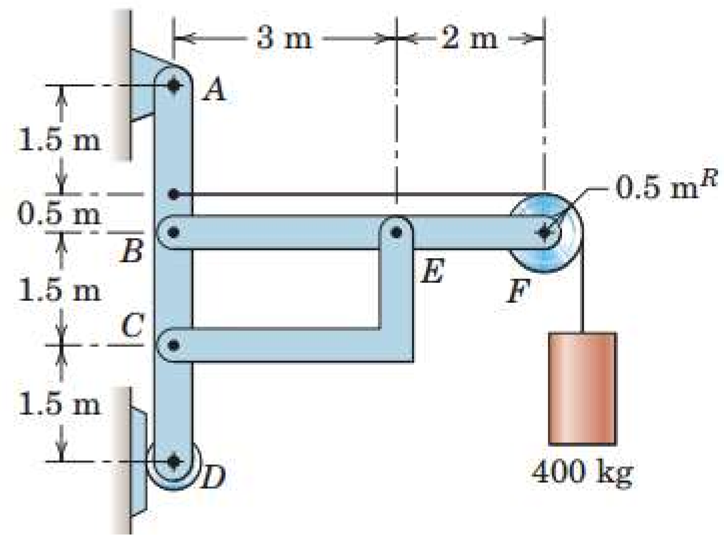
Mubarek Z

Frames and machines

- A structure is called a frame or machine if at least one of its individual members is a multiforce member. A multiforce member is defined as one with three or more forces acting on it, or one with two or more forces and one or more couples acting on it.
- Frames are structures which are designed to support applied loads and are usually fixed in position.
- Machines are structures which contain moving parts and are designed to transmit input forces or couples to output forces or couples. Because frames and machines contain multiforce members, the forces in these members in general will not be in the directions of the members. The forces acting on each member of a connected system are found by isolating the member with a free-body diagram and applying the equations of equilibrium. The principle of action and reaction must be carefully observed when we represent the forces of interaction on the separate free-body diagrams.
- If the structure contains more members or supports than are necessary to prevent collapse, then, as in the case of trusses, the problem is statically indeterminate, and the principles of equilibrium, although necessary, are not sufficient for solution. Although many frames and machines are statically indeterminate, we will consider in this article only those which are statically determinate.

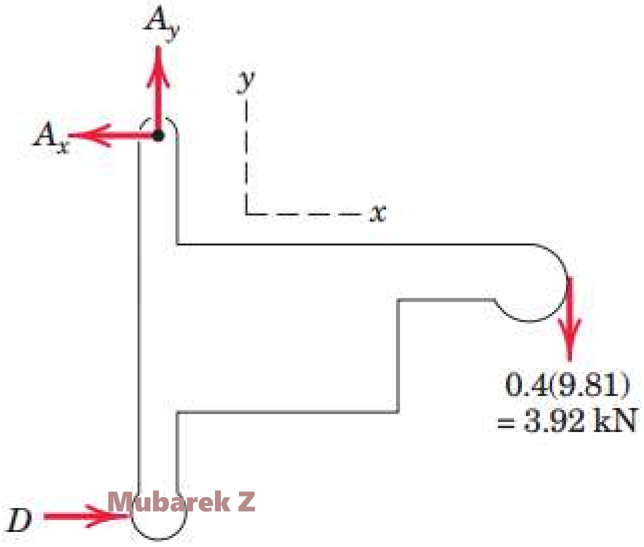
Frames and machines

Example: The frame supports the 400-kg load in the manner shown. Neglect the weights of the members compared with the forces induced by the load and compute the horizontal and vertical components of all forces acting on each of the members.



Solution:

From the FBD of the entire frame we determine the external reactions. Thus,



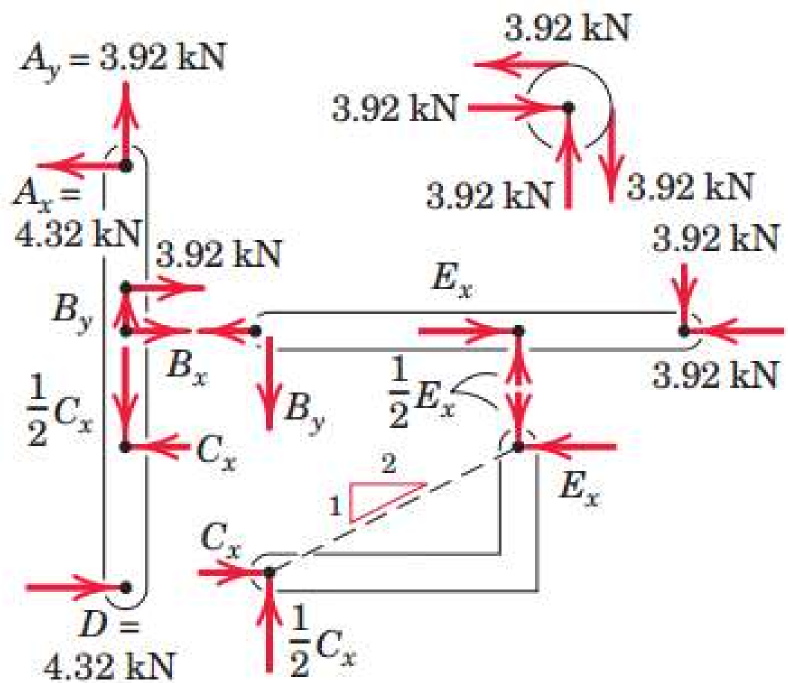
$[\Sigma M_A = 0]$	$5.5(0.4)(9.81) - 5D = 0$	$D = 4.32$ kN
$[\Sigma F_x = 0]$	$A_x - 4.32 = 0$	$A_x = 4.32$ kN
$[\Sigma F_y = 0]$	$A_y - 3.92 = 0$	$A_y = 3.92$ kN

Frames and machines

Solution: *continued…*

The solution may proceed by use of a moment equation about B or E for member BF, followed by the two force equations. Thus,

$[\Sigma M_B = 0]$	$3.92(5) - \frac{1}{2}E_x(3) = 0$	$E_x = 13.08$ kN
$[\Sigma F_y = 0]$	$B_y + 3.92 - 13.08/2 = 0$	$B_y = 2.62$ kN
$[\Sigma F_x = 0]$	$B_x + 3.92 - 13.08 = 0$	$B_x = 9.15$ kN



Positive numerical values of the unknowns mean that we assumed their directions correctly on the free-body diagrams. The value of Cx = Ex = 13.08 kN obtained by inspection of the free-body diagram of CE is now entered onto the diagram for AD, along with the values of Bx and By just determined. The equations of equilibrium may now be applied to member AD as a check, since all the forces acting on it have already been computed. The equations give

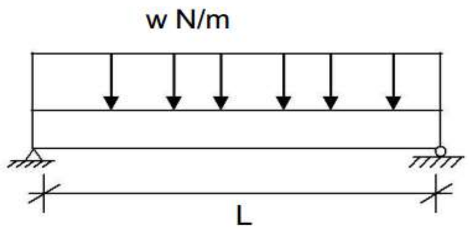
$[\Sigma M_C = 0]$	$4.32(3.5) + 4.32(1.5) - 3.92(2) - 9.15(1.5) = 0$
$[\Sigma F_x = 0]$	$4.32 - 13.08 + 9.15 + 3.92 + 4.32 = 0$
$[\Sigma F_y = 0]$	$-13.08/2 + 2.62 + 3.92 = 0$

Introduction

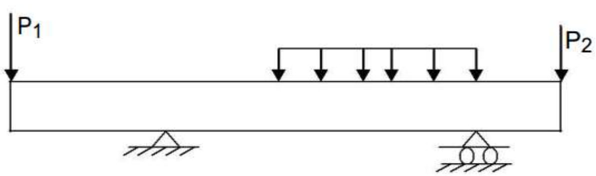
Beams are structural members which offer resistance to bending due to applied loads. Most beams are long prismatic bars, and the loads are usually applied normal to the axes of the bars.
Beams are undoubtedly the most important of all structural members, so it is important to understand the basic theory underlying their design.
To analyze the load-carrying capacities of a beam we must first establish the equilibrium requirements of the beam as a whole and any portion of it considered separately.
Second, we must establish the relations between the resulting forces and the accompanying internal resistance of the beam to support these forces. The first part of this analysis requires the application of the principles of statics. The second part involves the strength characteristics of the material and is usually treated in studies of the mechanics of solids or the mechanics of materials(i.e an other course).

Types of beams

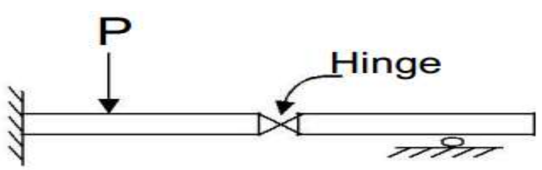
Beams supported so that their external support reactions can be calculated by the methods of statics alone are called statically determinate beams. A beam which has more supports than needed to provide equilibrium is statically indeterminate. To determine the support reactions for such a beam we must consider its load-deformation properties in addition to the equations of static equilibrium.



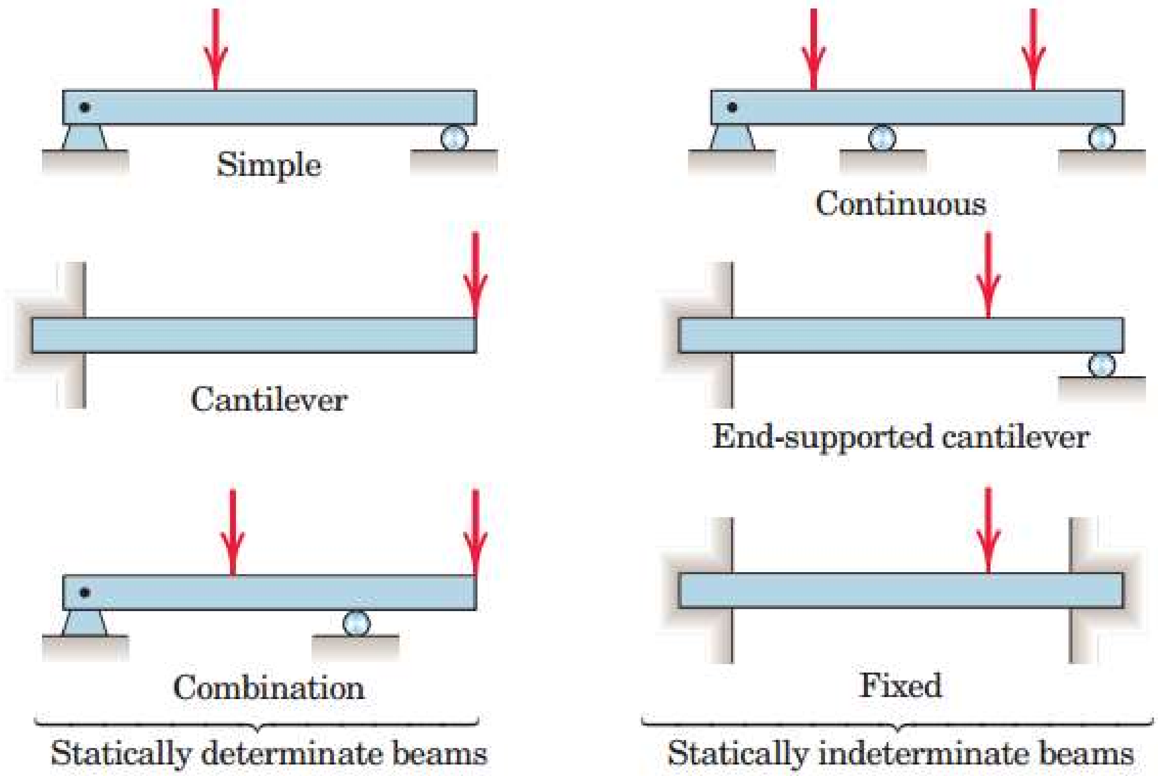
Simply supported beams (simple beams)



Overhanging beam



Compound beam (one example)

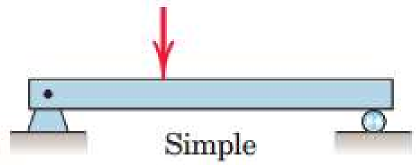


Determining external effects

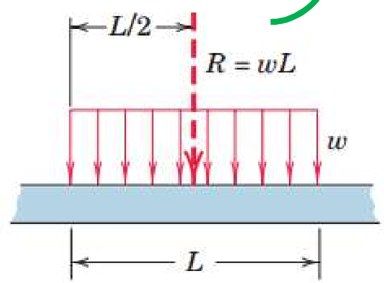
Beams may also be identified by the type of external loading they support.

Beams supporting

- **Concentrated loads**



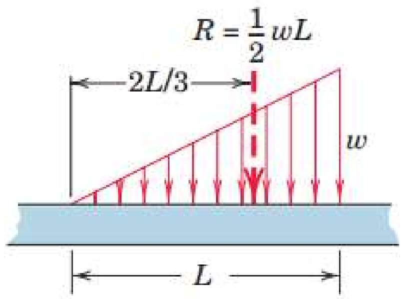
- **Distributed loads**
  - ✓ **Uniform: constant load intensity, w**



(a)

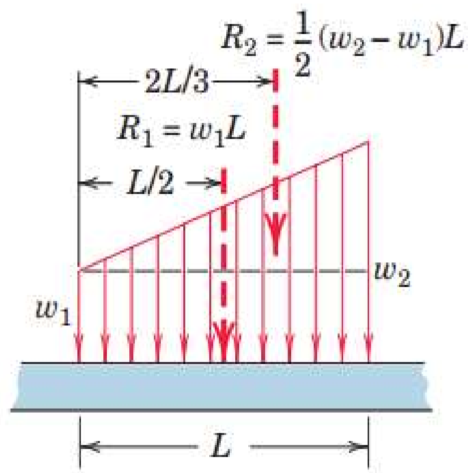
Resultant load R is represented by the area formed by the intensity w (force per unit length of beam) and the length L over which the force is distributed. The resultant passes through the centroid of this area.

  - ✓ **Linearly varying**



(b)

  - ✓ **Trapezoidal……………………**

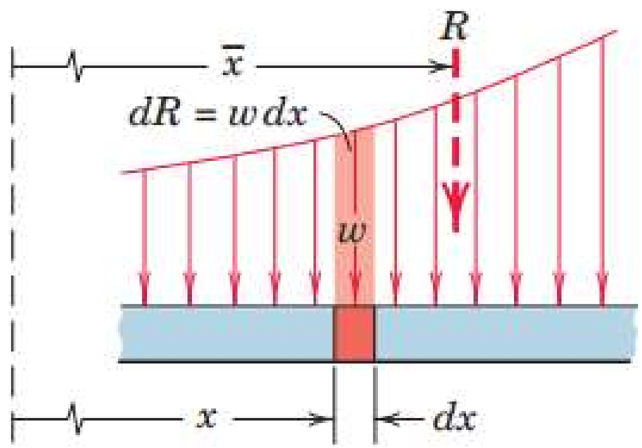


(c)

The trapezoidal area is broken into a rectangular and a triangular area, and the corresponding resultants R_1 and R_2 of these subareas are determined separately.

Determining external effects…

For a more general load distribution,



- **we must start with a differential increment of force dR =w dx. The total load R is then the sum of the differential forces, or**

$$R = \int w\,dx$$

- **The resultant R is located at the centroid of the area under consideration. The x-coordinate of this centroid is found by the principleof moments**

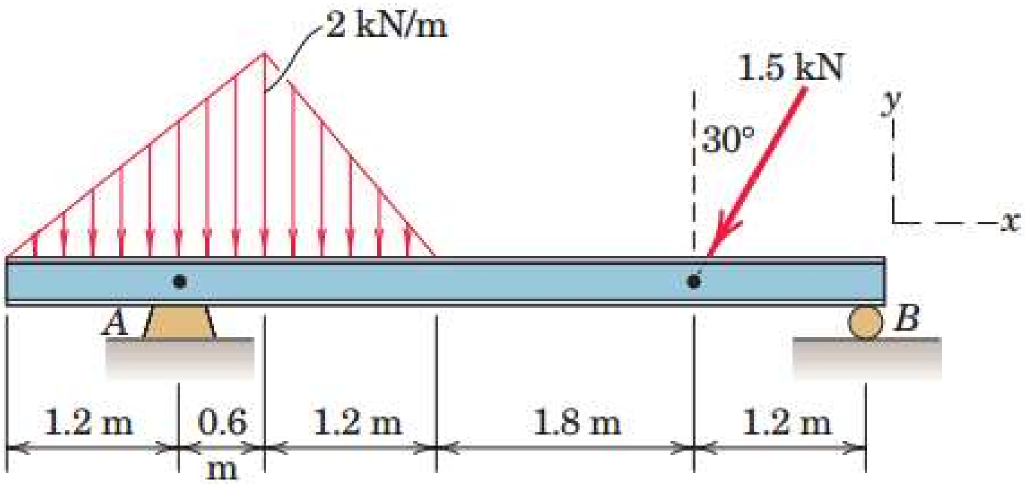
$$\bar{x} = \frac{\int xw\,dx}{R}$$

✓ **Once the distributed loads have been reduced to their equivalent concentrated loads, the external reactions acting on the beam may be found by a straightforward static analysis**

Determining external effects…

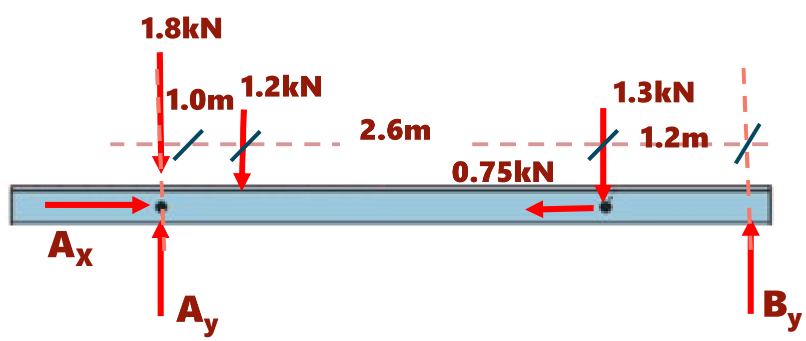
Example

Determine the reactions at A and B for the beam subjected to a combination of distributed and point loads.



$R_1=(2kN/m)*1.8m/2= 1.8kN$

$R_2=(2kN/m)*1.2m/2= 1.2kN$



$\Sigma M_A=0$, By*4.8 – 1.3*3.6 - 1.2*1 =0 ;
By=1.23 kN

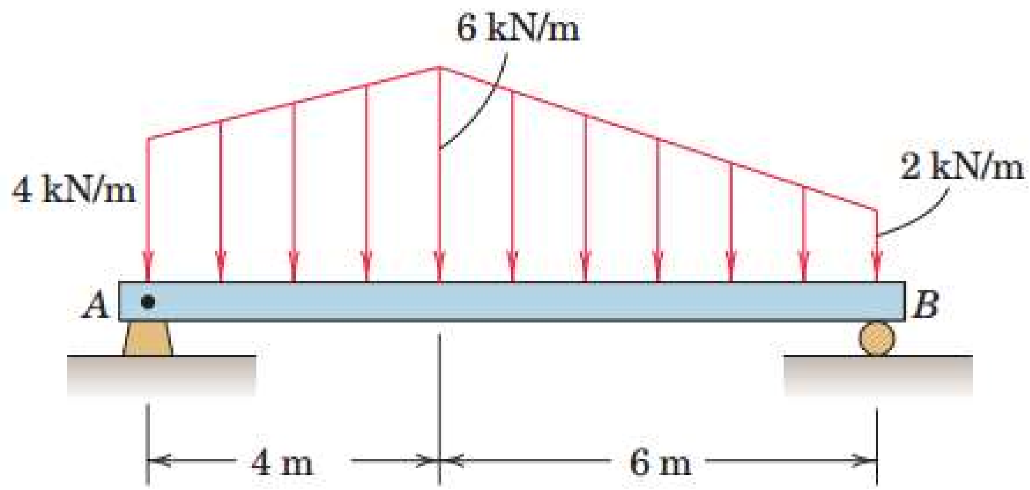
$\Sigma F_x=0$, Ax – 0.75=0;
Ax=0.75kN

$\Sigma F_y=0$, Ay + 1.23- 2 - 1.2 -1.3=0 ;
Ay=3.07 KN

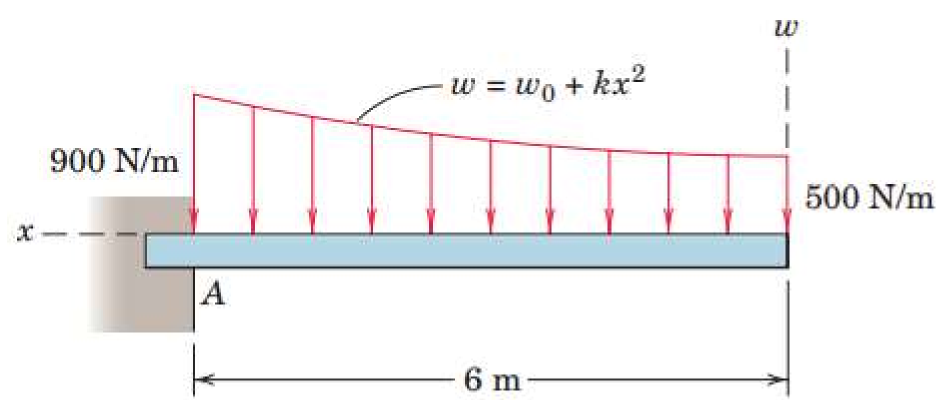
Determining external effects…

Exercise

1. Calculate the support reactions at A and B for the beam subjected to the two linearly varying load distributions



2. A cantilever beam supports the variable load shown. Calculate the supporting force R_A and moment M_A at A

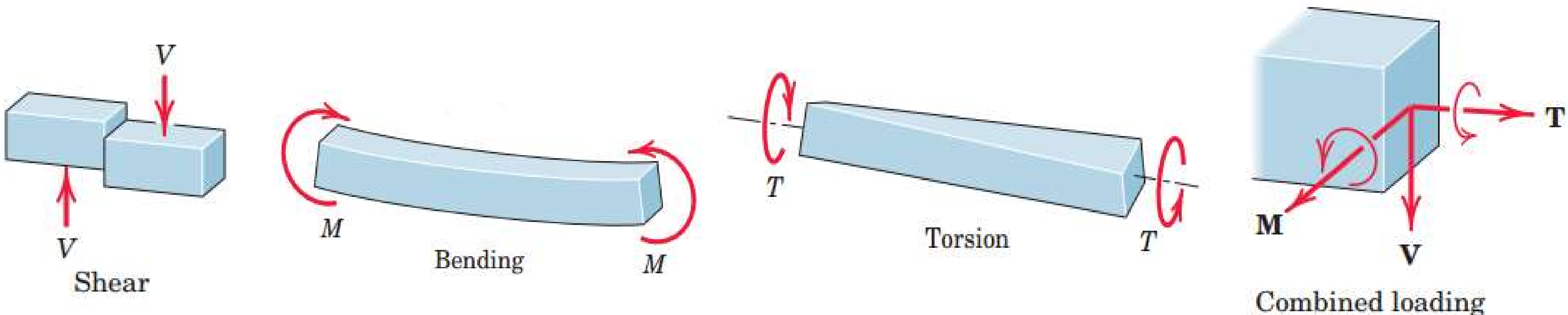


Determining internal effects

- **In this article we introduce internal beam effects and apply principles of statics to calculate the internal shear force and bending moment as functions of location along the beam.**

Shear, Bending, and Torsion

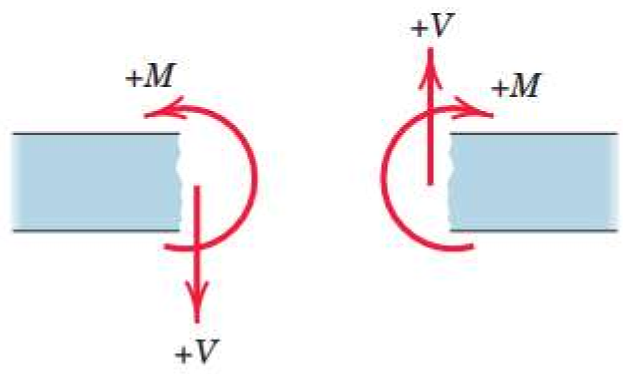
In addition to supporting tension or compression, a beam can resist shear, bending, and torsion. These three effects are illustrated in Figures below. The force V is called the shear force, the couple M is called the bending moment, and the couple T is called a torsional moment. These effects represent the vector components of the resultant of the forces acting on a transverse section of the beam as shown in the right figure.



Mubarek Z

Determining internal effects …

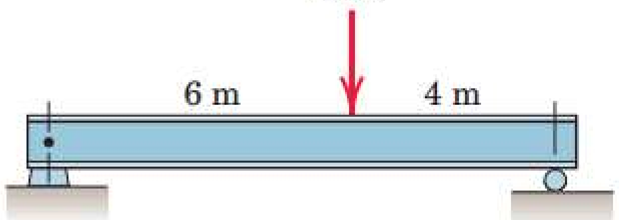
- Consider the shear force V and bending moment M caused by forces applied to the beam in a single plane. The conventions for positive values of shear V and bending moment M shown in Figure below are the ones generally used.
- From the principle of action and reaction we can see that the directions of V and M are reversed on the two sections. It is frequently impossible to tell without calculation whether the shear and moment at a particular section are positive or negative. For this reason it is advisable to represent V and M in their positive directions on the free-body diagrams and let the algebraic signs of the calculated values indicate the proper directions.



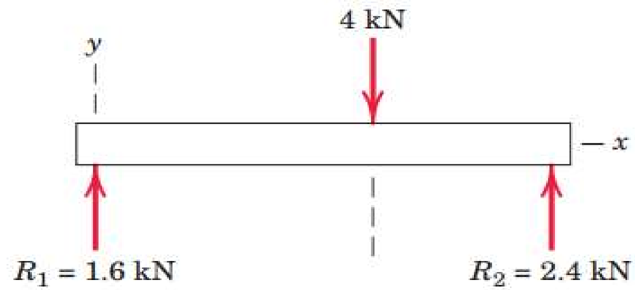
Determining internal effects …

Example

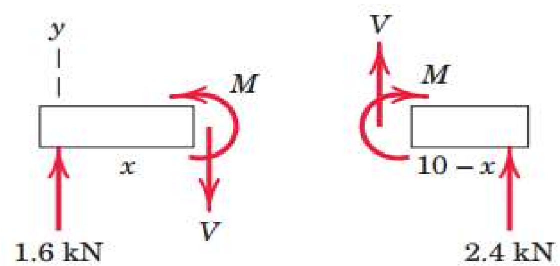
- **Determine the shear and moment distributions produced in the simple beam by the 4-kN concentrated load.**



Solution: 1st using FBD of entire system determine external or support reactions



A section of the beam of length x is next isolated with its FBD on which we show the shear V and the bending moment M in their positive directions. Equilibrium gives



$[\Sigma F_y = 0] \qquad V + 2.4 = 0 \qquad V = -2.4 \text{ kN}$

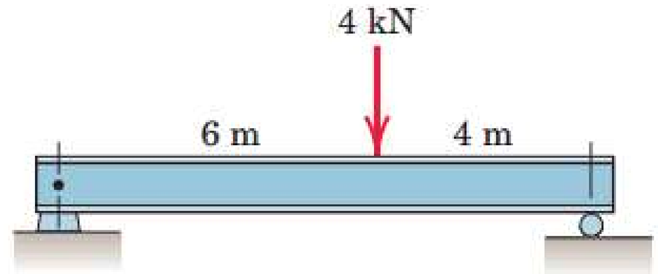
$[\Sigma M_{R_2} = 0] \qquad -(2.4)(10 - x) + M = 0 \qquad M = 2.4(10 - x)$

These values of V and M apply to all sections of the beam to the left of the 4-kN load. A section of the beam to the right of the 4-kN load is next isolated with its free-body diagram on which V and M are shown in their positive directions. Equilibrium requires

Shear and moment diagrams

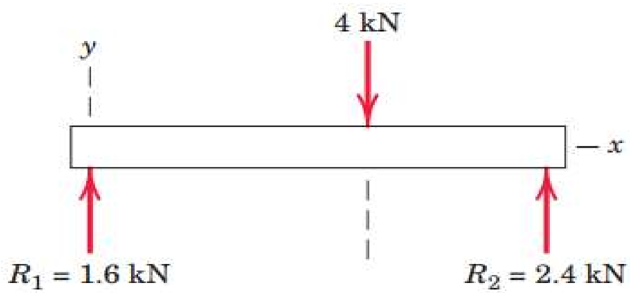
Example

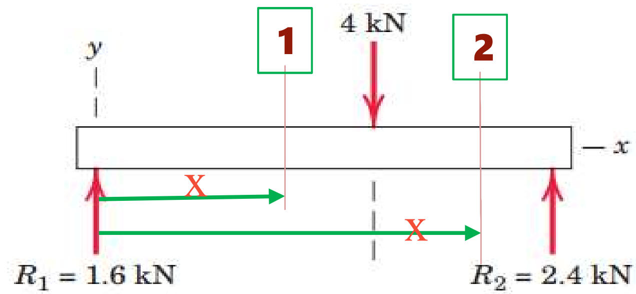
- **Determine the shear and moment distributions produced in the simple beam by the 4-kN concentrated load.**



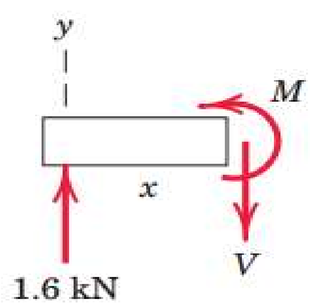
Solution:

1st using FBD of entire system determine external or support reactions





FBD of Section 1



A section of the beam of length x is next isolated with its FBD on which we show the shear V and the bending moment M in their positive directions. Equilibrium gives

$$[\Sigma F_y = 0] \qquad 1.6 - V = 0 \qquad V = 1.6 \text{ kN}$$

$$[\Sigma M_{R_1} = 0] \qquad M - 1.6x = 0 \qquad M = 1.6x$$

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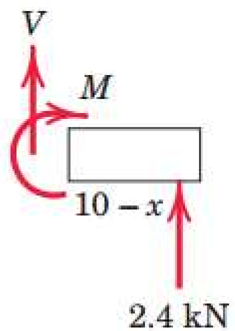
These values of V and M apply to all sections of the beam to the left of the 4-kN load.

Shear and moment diagrams…

Solution: ***continued…***

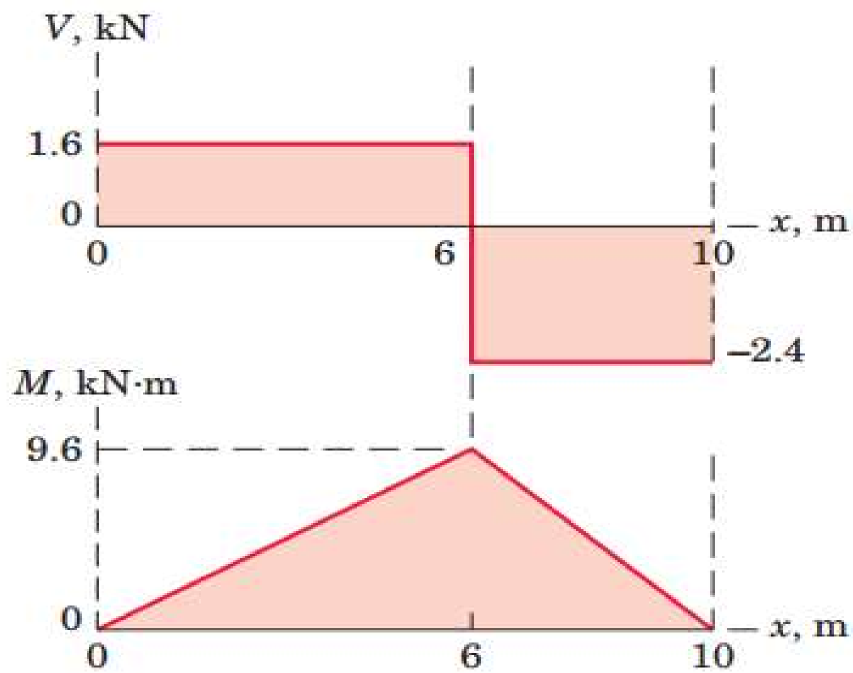
A section of the beam to the right of the 4-kN load is next isolated with its free-body diagram on which V and M are shown in their positive directions. Equilibrium requires

$[\Sigma F_y = 0]$	$V + 2.4 = 0$	$V = -2.4\ \text{kN}$
$[\Sigma M_{R_2} = 0]$	$-(2.4)(10 - x) + M = 0$	$M = 2.4(10 - x)$



FBD of Section 2

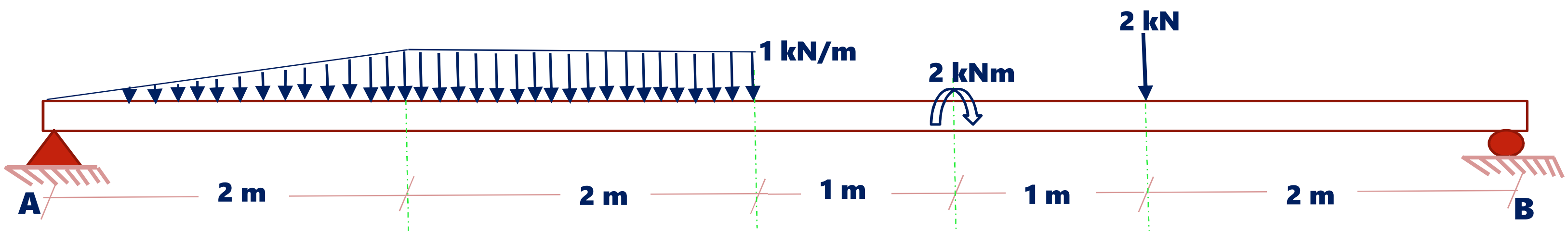
These results apply only to sections of the beam to the right of the 4-kN load. The values of V and M are plotted as shown.



The maximum bending moment occurs where the shear changes direction. As we move in the positive x-direction starting with x = 0, we see that the moment M is merely the accumulated area under the shear diagram.

Shear and moment diagrams…

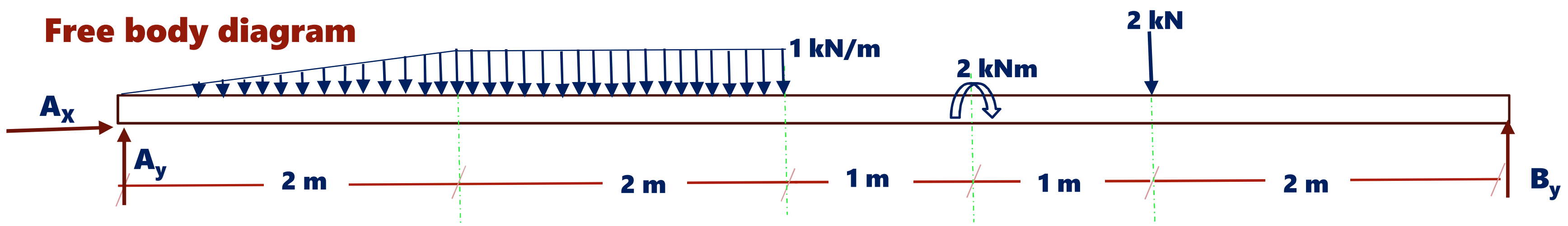
Example 2: Construct the shear and moment diagrams for the beam loaded as shown



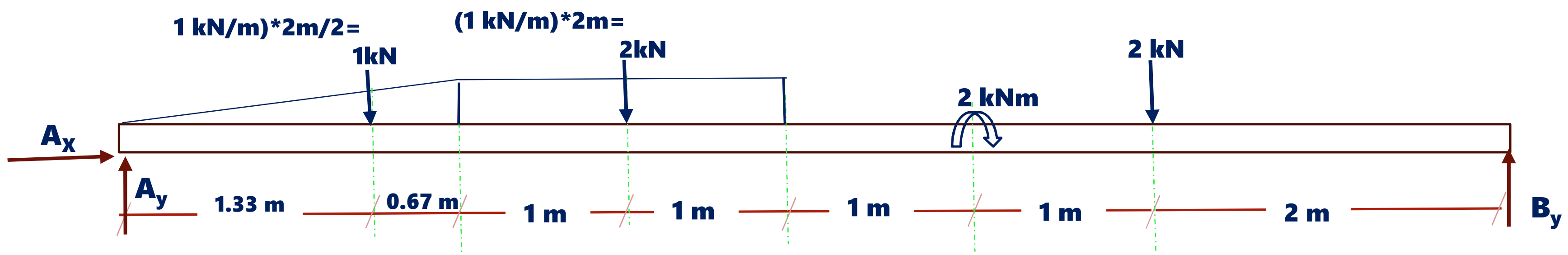
Solution:

i) Analyze the beam externally (i.e determine support reactions)

- **Free body diagram**



- **Substitute the distributed force by its equivalent concentrated force at its centroid**



By considering the beam as it is under state of equilibrium, we can solve the unknowns through equations of equilibrium

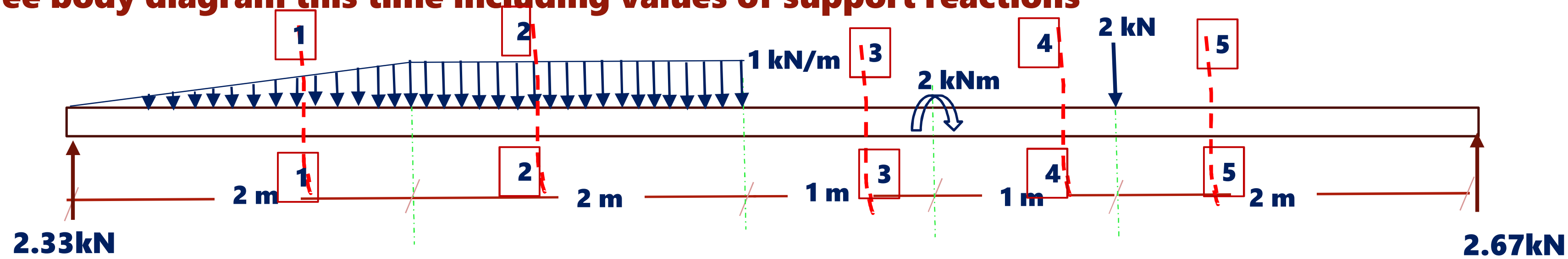
Shear and bending moment diagrams…

Example 2: Solution…

- ✓ ∑Fx=0, Assume ⟶ as positive, hence Ax+0=0, Ax=0
- ✓ $\sum M_A$=0, Assume CCW as positive moment, Thus
  By*8m-1kN*1.33m-2kN*3m-2kNm-2kN*6m=0; By=2.67kN (↑)
- ✓ ∑Fy=0, Assume ↑ as positive, thus Ay+By-1kN-2kN-2kN=0,
  Ay+By=5kN
  by substituting the value of By=2.67 kN, then Ay=2.33kN (↑)

ii) Analyze the beam internally (i.e determine internal actions such as shear and bending moment)

- Free body diagram this time including values of support reactions



- Decide on the number of segments to be analyzed. (i.e the number of sections to cut the beam)
- Show the representative sections on the above free body diagram

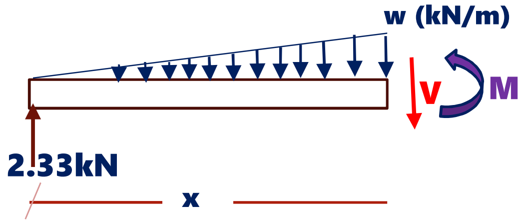
(0≤ X ≤2)…by section 1---1 where x: distance on the beam as it is measured from left end.
(2≤X ≤4)…by section 2---2
(4≤ X ≤5)…by section 3---3
(5≤ X ≤6)…by section 4---4
(6≤ X ≤8)…by section 5---5

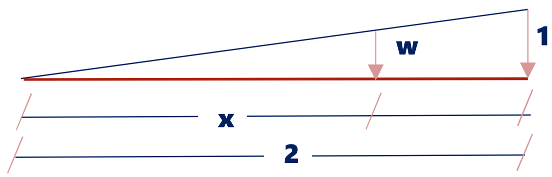
Shear and moment diagrams…

Example 2: solution …

- **Analysis of beam segment ($0 < X \leq 2$)**

- **Free body diagram of LHS of section 1---1**

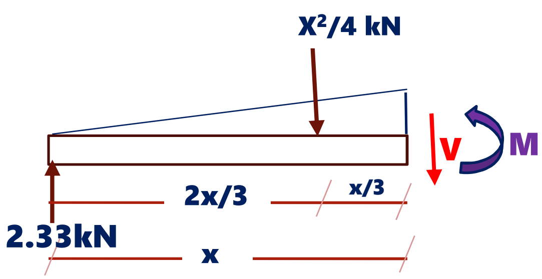




- From triangle similarity, $w/1 = x/2$; Thus $w = (x/2)$kN/m.
- Its equivalent concentrated force $= w*x/2$

  $= (x/2)*(x/2) = x^2/4$

  Which is to be located $2x/3$ from left end



- Now it is easy to write equilibrium equation
- ✓ From

  $\sum Fy = 0$, Assume ↑ as positive, thus

  $2.33 - x^2/4 - V = 0$, $V = 2.33 - x^2/4$; When $x = 0$, $V = 2.33$ kN

  When $X = 2$, $V = 2.33 - 2^2/4 = 1.33$kN

  $\sum M_A = 0$, Assume CCW as positive moment, thus

  $M - 2.33*x + (x^2/4)*x/3 = 0$; $M = 2.33x - x^3/12$

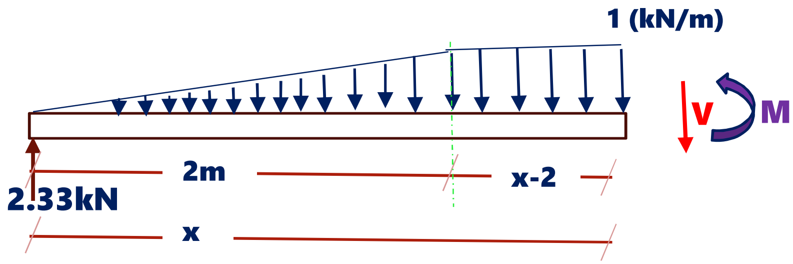
  When $x = 0$, $M = 2.33*0 - 0^3/12 = 0$

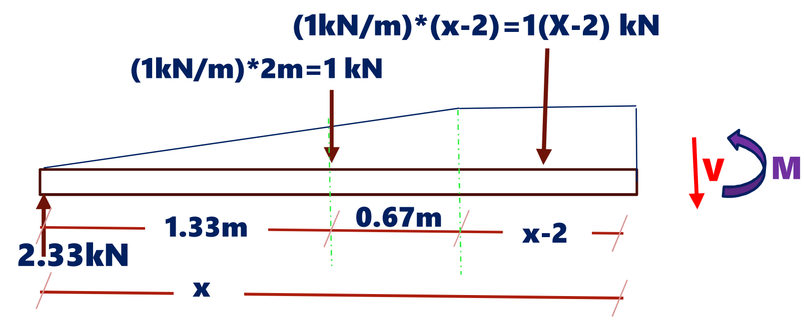
  When $X = 2$, $M = 2.33*2 - 2^3/12 = 4$kN

Shear and moment diagrams…

Example 2: solution …

- **Analysis of beam segment (2≤ X ≤4)**
  - Free body diagram of LHS of section 2---2





- **Now it is easy to write equilibrium equation**

✓ **From**

∑Fy=0, Assume ↑ as positive, thus

2.33-1-1(x-2)-V=0, V=2.33-1-(x-2); V=1.33 - (x-2);

When x=2, V=1.33 kN
When X=4, V=1.33 - (4-2)=-0.67kN

$\sum M_A$=0, Assume CCW as positive moment, thus

M-2.33*x+1*(x-1.33)+1(x-2)*(x-2)/2=0;
M=2.33x-(x-1.33)-(x-2)*(x-2)/2

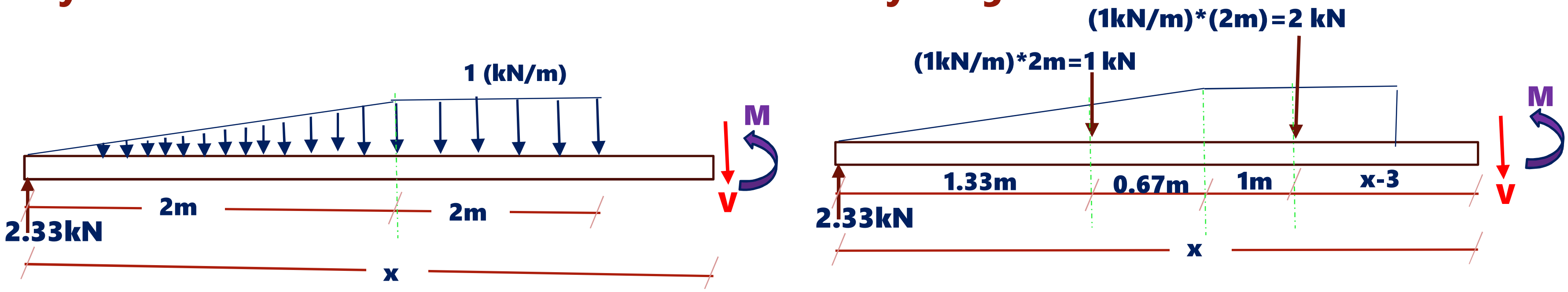
When x=2, M=2.33*2-(2-1.33)-(2-2)*(2-2)/2=4kNm
When X=4, M=2.33*4-(4-1.33)-(4-2)*(4-2)/2=4.65kNm

Shear and moment diagrams…

Example 2: solution …

- **Analysis of beam segment ($4 \leq X < 5$)**

- Free body diagram of LHS of section 3---3. RHS of this and other sections afterwards may be easy. But lets use LHS for all of them. You can check by using RHS



- Now it is easy to write equilibrium equation

✓ From

$\sum Fy=0$, Assume ↑ as positive, thus

$2.33-1-2-V=0$, $V=2.33-1-2=-0.67kN$

(Shear force is constant throughout this segment);

When $x=4$, $V=-0.67$ kN

When $X=5$, $V=-0.67kN$

$\sum M_A=0$, Assume CCW as positive moment, thus

$M-2.33*x+1*(x-1.33)+2(x-3)=0;$

$M=2.33x-(x-1.33)-2(x-3)$

When $x=4$, $M=2.33*4-(4-1.33)-2(4-3)=4.65kNm$

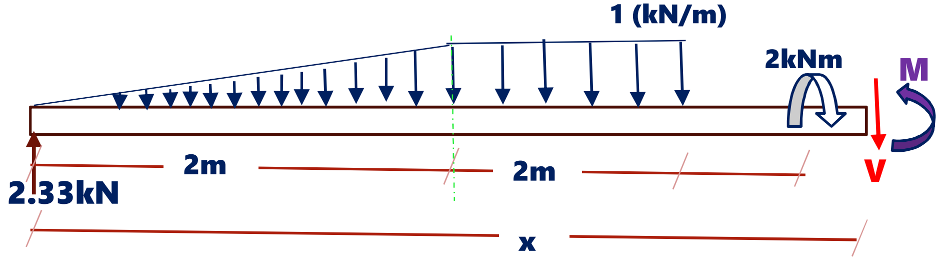
When $X=5$, $M=2.33*5-(5-1.33)-2(5-3)=3.98kNm$

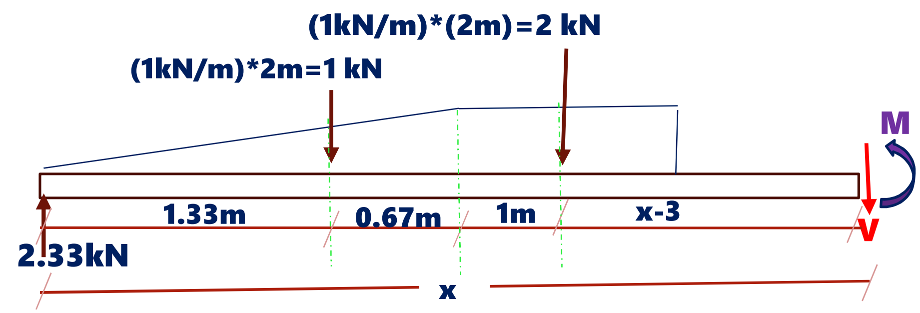
Shear and moment diagrams…

Example 2: solution …

- **Analysis of beam segment (5< X <6)**

  - **Free body diagram of LHS of section 4---4.**





- **Now using equilibrium equation**
- ✓ **From**

∑Fy=0, Assume ↑ as positive, thus

2.33-1-2-V=0, V=2.33-1-2=-0.67kN

(Shear force is constant throughout this segment);

When x=4, V=-0.67 kN

When X=5, V=-0.67kN

$\sum M_A$=0, Assume CCW as positive moment, thus

M-2.33*x+1*(x-1.33)+2(x-3)-2=0;

M=2.33x-(x-1.33)-2(x-3)+2

When x=5, M=2.33*5-(5-1.33)-2(5-3)+2=5.98kNm

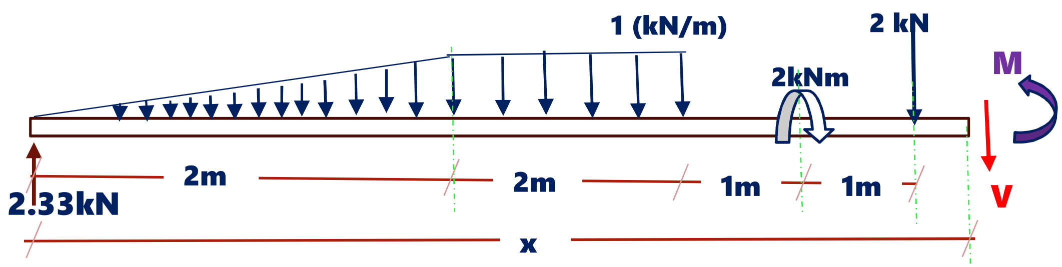
When X=6, M=2.33*6-(6-1.33)-2(6-3)+2=5.31kNm

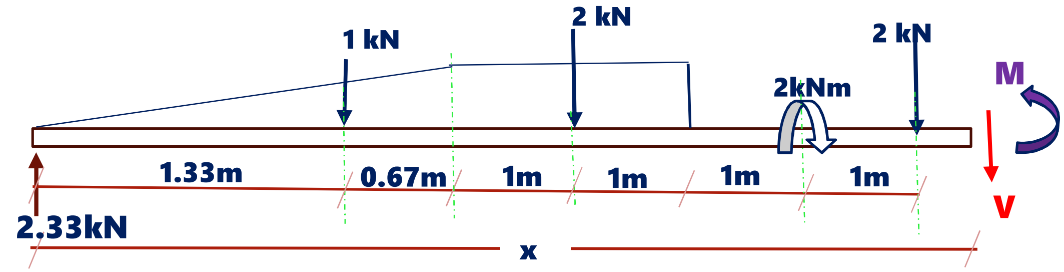
Shear and moment diagrams…

Example 2: solution …

➢ Analysis of beam segment (6< X <8)

- Free body diagram of LHS of section 5---5





- Now using equilibrium equation

✓ From $\sum Fy=0$, Assume ↑ as positive, thus 2.33-1-2-2-V=0, V=2.33-1-2-2=-2.67kN (Shear force is constant throughout this segment);

When x=6, V=-2.67 kN

When X=8, V=-2.67kN

$\sum M_A=0$, Assume CCW as positive moment, thus

M-2.33*x+1*(x-1.33)+2(x-3)-2+2(x-6)=0;

M=2.33x-(x-1.33)-2(x-3)-2(x-6)+2

When x=6, M=2.33*6-(6-1.33)-2(6-3)-2(6-6)+2=5.31kNm

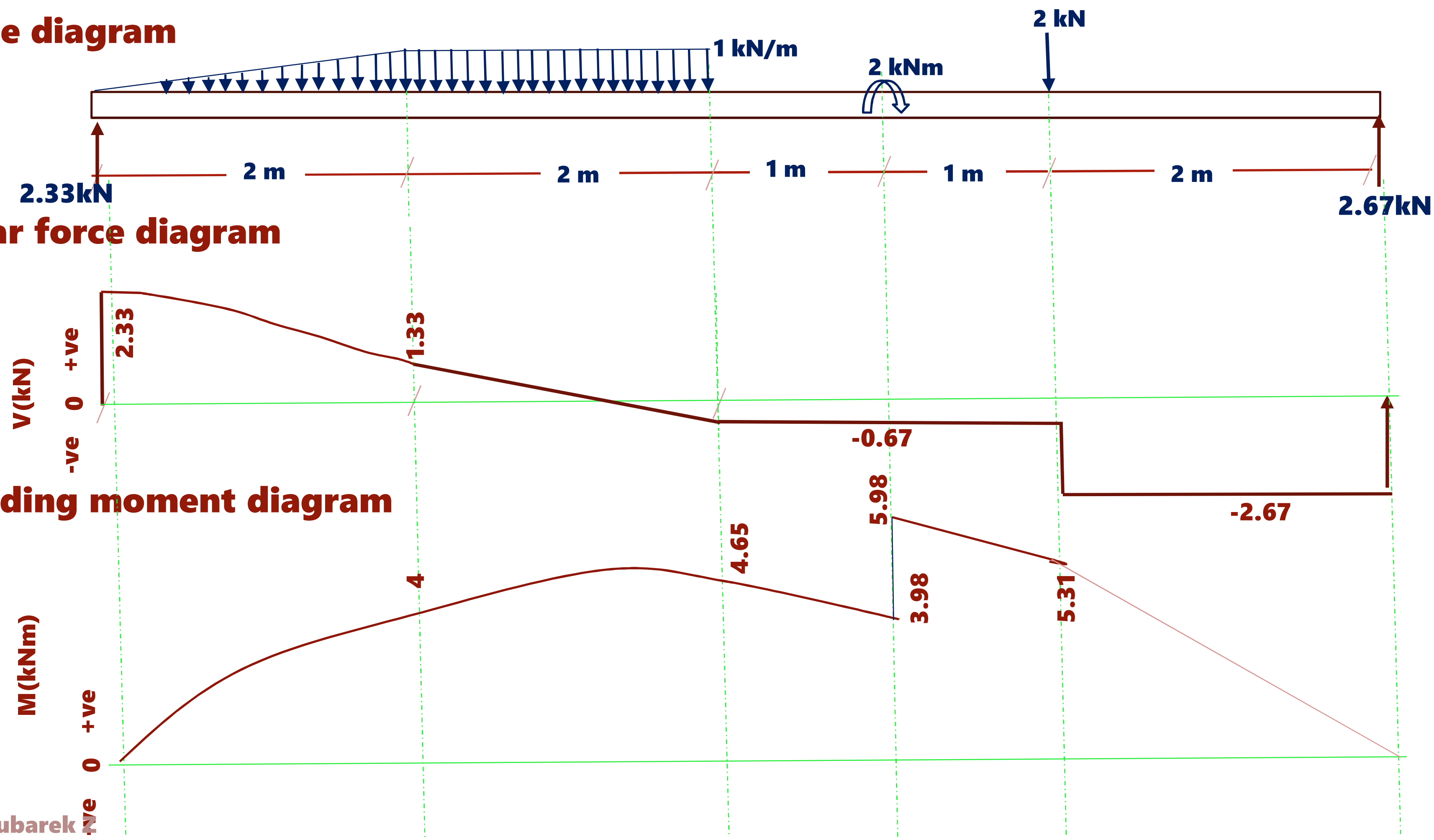
When X=8, M=2.33*8-(8-1.33)-2(8-3)-2(8-6)+2=0kNm

Shear and moment diagrams…

Example 2: solution …

iii) Draw diagrams

- Force diagram
- Shear force diagram
- Bending moment diagram



Shear and moment diagrams…

Example 2: Solution …

iv) Determine the minimum and maximum moment

The location of minimum and maximum moment corresponds to the location where shear force is zero. Thus, look at the shear force diagram and identify the segment. After this use the segment`s equation of shear to be equal to zero.

i.e in this case segment two($2 \le x \le 4$) with equations

- ✓ $V=1.33-(x-2)$
- ✓ $M=2.33x-(x-1.33)-(x-2)*(x-2)/2$

@ $M_{max,}$, $V=0$ hence $1.33-(x-2)=0$, $x=3.33m$ from left is the location Where shear force is zero and corresponding maximum moment in this segement.

Substituting $x=3.33m$ in the moment equation gives the maximum moment as $M_{max}=2.33*3.33-(3.33-1.33)-(3.33-2)*(3.33-2)/2$

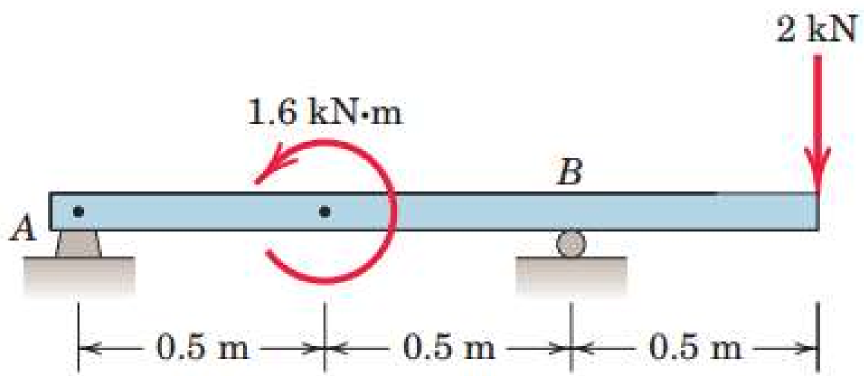
$=7.76-2-0.88$; $M_{max}=4.88kNm$ in segment two. But look at the BMD, due to presence of CW concentrated moment at X=3, The BMD rises by that amount resulting in the maximum moment through out the beam. Thus $M_{max}=5.98$ kNm

$M_{min}=0$ by inspection, at the both in ends of the beam

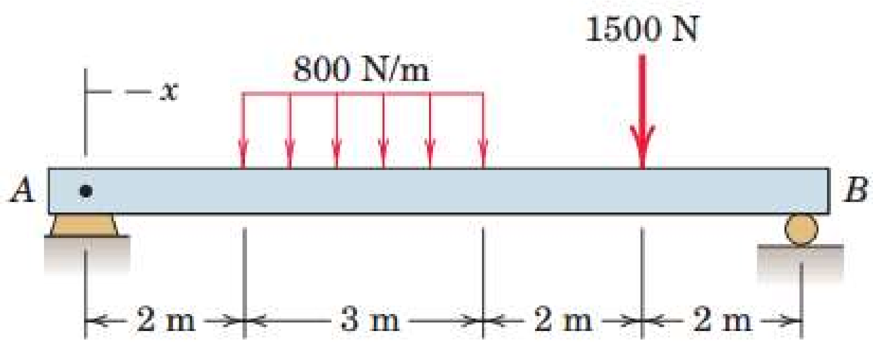
Shear and moment diagrams…

Exercise

1. Construct the shear and moment diagrams for the beam loaded by the 2-kN force and the 1.6-kN m couple. State the value of the bending moment at point B.

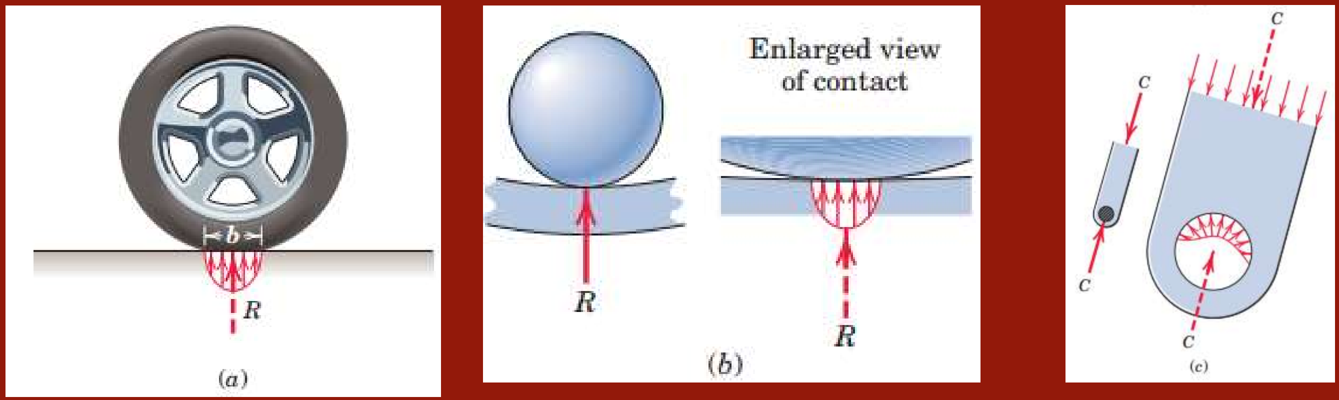


2. Plot the shear and moment diagrams for the beam loaded with both the distributed and point loads. What are the values of the shear and moment at Determine the maximum bending moment, M_{max}



Introduction

Actually, “concentrated” forces do not exist in the exact sense, since every external force applied mechanically to a body is distributed over a finite contact area however small.



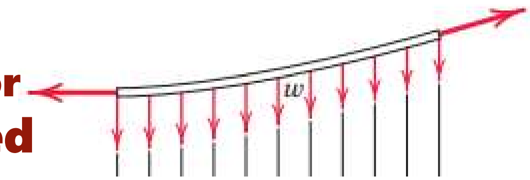
When analyzing the forces acting on the car as a whole, if the dimension b of the contact area is negligible compared with the other pertinent dimensions, such as the distance between wheels, then we may replace the actual distributed contact forces by their resultant R treated as a concentrated force.

- In this and other similar examples we may treat the forces as concentrated when analyzing their external effects on bodies as a whole.
- If, on the other hand, we want to find the distribution of internal forces in the material of the body near the contact location, where the internal stresses and strains may be appreciable, then we must not treat the load as concentrated but must consider the actual distribution. This king of problem will not be discussed in this course because it requires a knowledge of the properties of the material and belongs in more advanced treatments of the mechanics of materials and the theories of elasticity and plasticity.

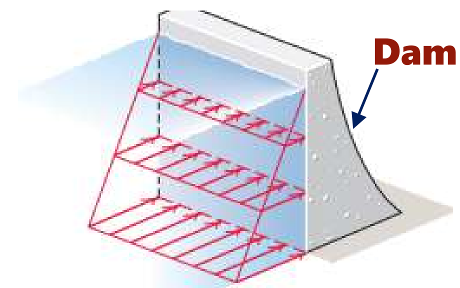
Introduction

When forces are applied over a region whose dimensions are not negligible compared with other pertinent dimensions, then we must account for the actual manner in which the force is distributed by summing up the effects of the distributed force over the entire region. We carry out this process by using the procedures of mathematical integration.
For this purpose we need to know the intensity of the force at any location. There are three categories into which such problems fall;

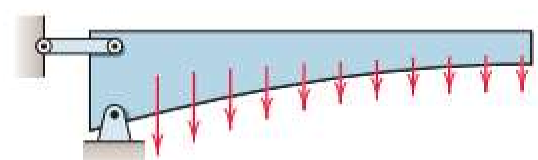
1) Line Distribution :- when a force is distributed along a line. The loading is expressed as force per unit length of line(N/m or kN/m). For example the continuous vertical load supported by a suspended cable, Fig. *a*, the intensity *w* of the loading is expressed as force per unit length of line.



2) Area Distribution :- when a force is distributed over an area. The loading is expressed as force per unit area (N/m^2).



3)Volume Distribution :- when a force is distributed over the volume of a body (body force), (N/m^3). The body force due to the earth`s gravitational attraction (weight) is by far the most commonly encountered distributed force. The intensity of gravitational force is the *specific weight ϱg*, where *ϱ* is the density (mass per unit volume, kg/m^3) and *g* is the acceleration due to gravity(m/s^2).



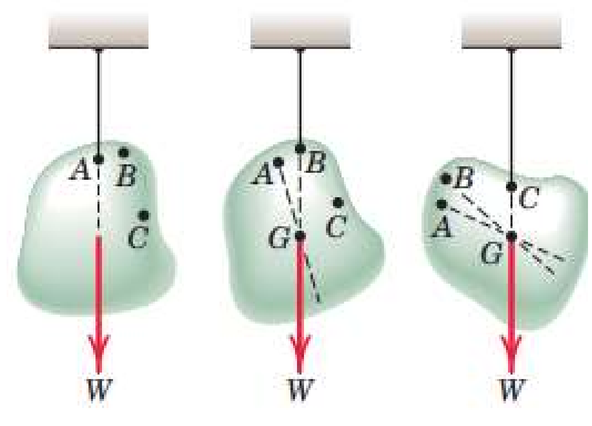
(c)

Center of Mass Vs. Centre of gravity

Center of Mass

Consider a three-dimensional body of any size and shape, having a mass m. If we suspend the body, as shown in Figure below, from any point such as A, the body will be in equilibrium under the action of the tension in the cord and the resultant W of the gravitational forces acting on all particles of the body. This resultant is clearly collinear with the cord. Assume that we mark its position by drilling a hypothetical hole of negligible size along its line of action. We repeat the experiment by suspending the body from other points such as B and C, and in each instance we mark the line of action of the resultant force. For all practical purposes these lines of action will be concurrent at a single point G, which is called the center of gravity of the body.

An exact analysis, however, would account for the slightly differing directions of the gravity forces for the various particles of the body, because those forces converge toward the center of attraction of the earth.
Also, because the particles are at different distances from the earth, the intensity of the force field of the earth is not exactly constant over the body. As a result, the lines of action of the gravity-force resultants in the experiments just described will not be quite concurrent, and therefore no unique center of gravity exists in the exact sense. This is of no practical importance as long as we deal with bodies whose dimensions are small compared with those of the earth. We therefore assume a uniform and parallel force field due to the gravitational attraction of the earth, and this assumption results in the concept of a unique center of gravity.

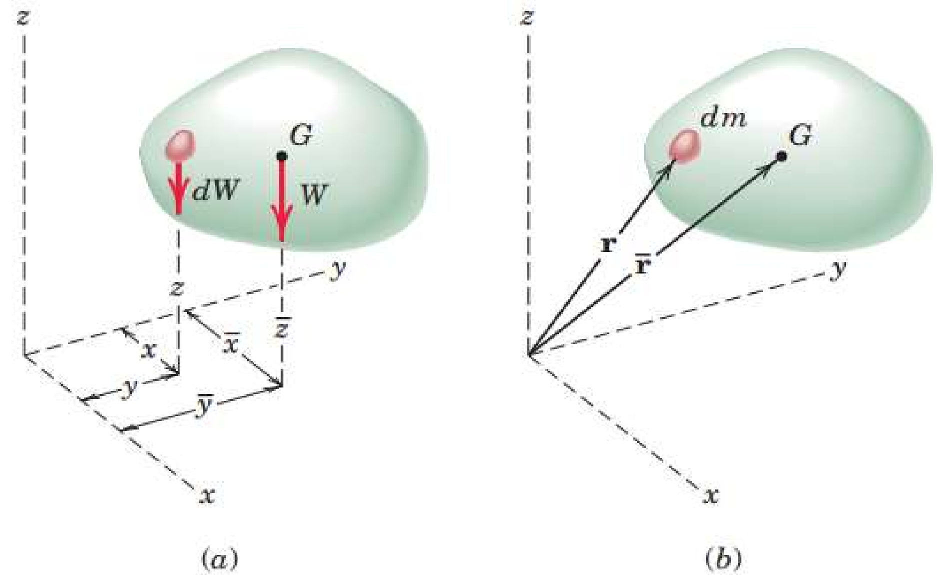


(a) (b) (c)

Center of Mass Vs. Centre of gravity…

Determining the Center of Gravity

To determine mathematically the location of the center of gravity of any body, Fig. a, we apply the principle of moments to the parallel system of gravitational forces.



we may express the coordinates of the center of gravity G as

$$\bar{x} = \frac{\int x \, dW}{W} \qquad \bar{y} = \frac{\int y \, dW}{W} \qquad \bar{z} = \frac{\int z \, dW}{W}$$

$$W = \int dW.$$

$$W = mg$$

With the substitution of W=mg and dW=gdm, the expressions for the coordinates of the center of gravity become

$$\bar{x} = \frac{\int x \, dm}{m} \qquad \bar{y} = \frac{\int y \, dm}{m} \qquad \bar{z} = \frac{\int z \, dm}{m} \qquad \textbf{(equ.1)}$$

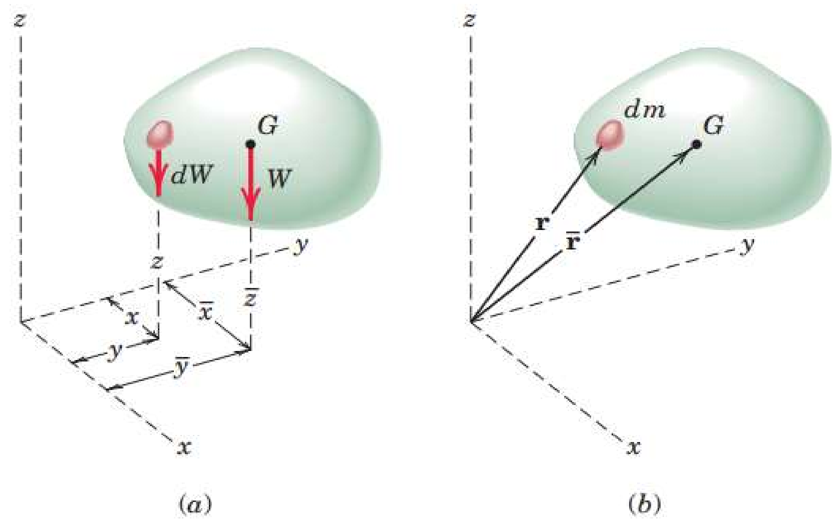
Center of Mass Vs. Centre of gravity…

Determining the Center of Gravity…

The equations above may be expressed in vector form with the aid of Fig. (b), in which the elemental mass and the mass center G are located by heir respective position vectors as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
$$\bar{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$$

$$\bar{\mathbf{r}} = \frac{\int \mathbf{r}\, dm}{m} \quad \text{(equ.2)}$$



The density ρ of a body is its mass per unit volume. Thus, the mass of a differential element of volume dV becomes dm =ρ dV.

$$\bar{x} = \frac{\int x\rho\, dV}{\int \rho\, dV} \qquad \bar{y} = \frac{\int y\rho\, dV}{\int \rho\, dV} \qquad \bar{z} = \frac{\int z\rho\, dV}{\int \rho\, dV} \quad \text{(equ.3)}$$

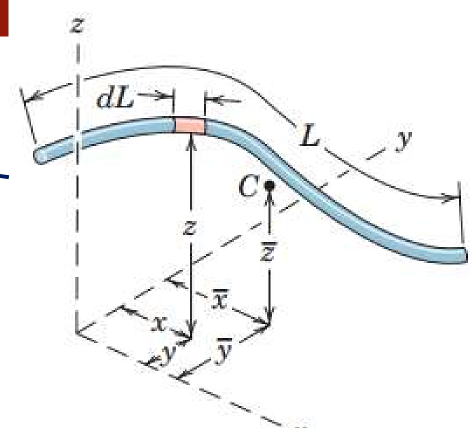
Center of Mass Vs. Centre of gravity…

In summary…

- As it can be seen from the above equations, the formulas are independent of gravitational effects since g no longer appears. They therefore define a unique point in the body which is a function solely of the distribution of mass. This point is called the center of mass, and clearly it coincides with the center of gravity as long as the gravity field is treated as uniform and parallel.
- It is meaningless to speak of the center of gravity of a body which is removed from the gravitational field of the earth, since no gravitational forces would act on it. The body would, however, still have its unique center of mass. We will usually refer henceforth to the center of mass rather than to the center of gravity. Also, the center of mass has a special significance in calculating the dynamic response of a body to unbalanced forces.
- In most problems the calculation of the position of the center of mass may be simplified by an intelligent choice of reference axes. In general the axes should be placed so as to simplify the equations of the boundaries as much as possible. Thus, polar coordinates will be useful for bodies with circular boundaries. Another important clue may be taken from considerations of symmetry. Whenever there exists a line or plane of symmetry in a homogeneous body, a coordinate axis or plane should be chosen to coincide with this line or plane. The center of mass will always lie on such a line or plane, since the moments due to symmetrically located elements will always cancel, and the body may be considered composed of pairs of these elements.

Centroids of Lines, Areas, and Volumes

When the density of a body is uniform throughout, it will be a constant factor in both the numerators and denominators of Equ. 3 above and will therefore cancel. The remaining expressions define a purely geometrical property of the body, since any reference to its mass properties has disappeared. The term centroid is used when the calculation concerns a geometrical shape only. When speaking of an actual physical body, we use the term center of mass. If the density is uniform throughout the body, the positions of the centroid and center of mass are identical, whereas if the density varies, these two points will, in general, not coincide

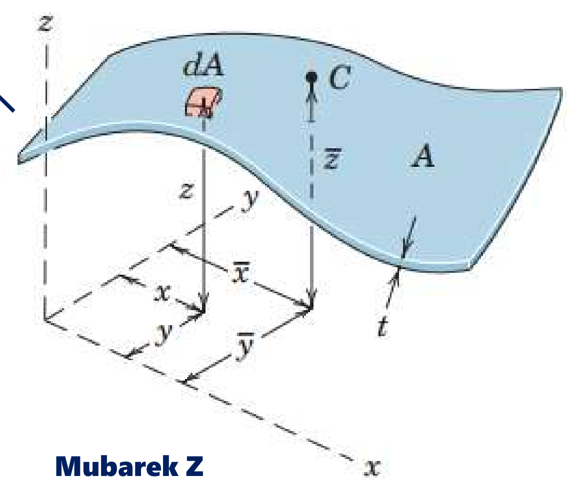


(1) Lines

$$\bar{x} = \frac{\int x\, dL}{L} \qquad \bar{y} = \frac{\int y\, dL}{L} \qquad \bar{z} = \frac{\int z\, dL}{L}$$

(2) Areas

$$\bar{x} = \frac{\int x\, dA}{A} \qquad \bar{y} = \frac{\int y\, dA}{A} \qquad \bar{z} = \frac{\int z\, dA}{A}$$



(3) Volumes.

$$\bar{x} = \frac{\int x\, dV}{V} \qquad \bar{y} = \frac{\int y\, dV}{V} \qquad \bar{z} = \frac{\int z\, dV}{V}$$

$(\bar{x}, \bar{y}, \bar{z})$=the coordinates of the center of mass also become the coordinates of the centroid C of the body

Mubarek Z

Centroids of Lines, Areas, and Volumes…

Choice of Element for Integration

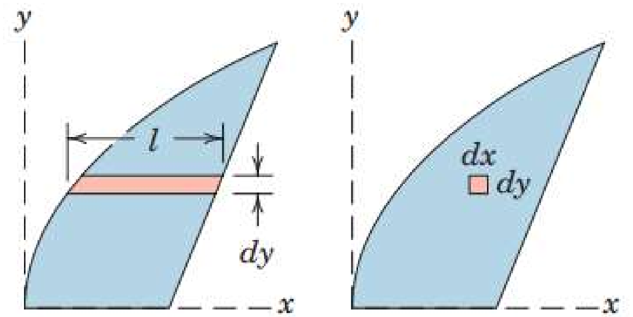
✓ **With mass centers and centroids the concept of the moment principle is simple enough; the difficult steps are the choice of the differential element and setting up the integrals. The following five guidelines will be useful.**

1. Order of Element

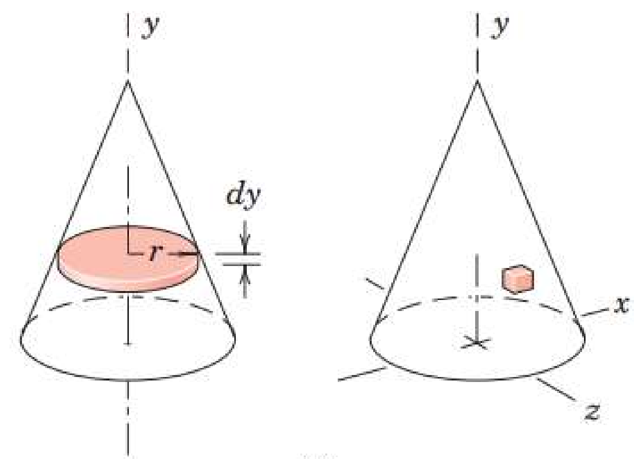
Whenever possible, a first-order differential element should be selected in preference to a higher-order element so that only one integration will be required to cover the entire figure

Eg. For Figure (a) a first-order horizontal strip of area dA = l dy will require only one integration with respect to y to cover the entire figure. Where as the second-order element dx dy will require two integrations, first with respect to x and second with respect to y, to cover the figure.

For the solid cone in Figure (*b*) we choose a first-order element in the form of a circular slice of volume *dV*= $\pi r^2 dy$. This choice requires only one integration, and thus is preferable to choosing a third-order element *dV = dx dy dz*, which would require three awkward integrations.



(*a*)



(*b*)

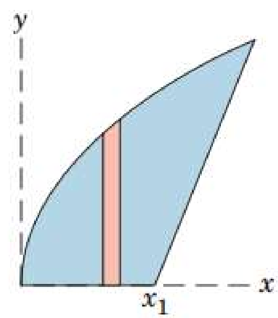
Centroids of Lines, Areas, and Volumes…

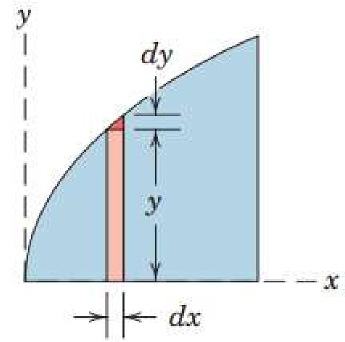
Choice of Element for Integration…

2. Continuity

Whenever possible, we choose an element which can be integrated in one continuous operation to cover the figure. Thus, the horizontal strip in Figure below would be preferable to the vertical strip which would require two separate integrals because of the discontinuity in the expression for the height of the strip at x = x1.

3. Discarding Higher-Order Terms.

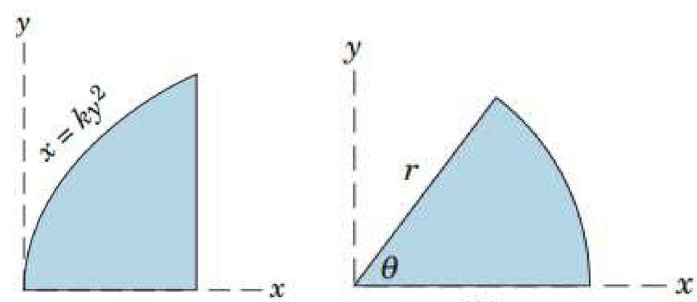




Higher-order terms may always be dropped compared with lower-order terms.
Thus, the vertical strip of area under the curve in Figure shown to the right is given by the first-order term $dA = y\,dx$, and the second-order triangular area $dx\,dy/2$ is discarded. In the limit, of course, there is no error

4. Choice of Coordinates.

As a general rule, we choose the coordinate system which best matches the boundaries of the figure. Thus, the boundaries of the area in Fig. (a) are most easily described in rectangular coordinates, whereas the boundaries of the circular sectors of Fig. (b) are best suited to polar coordinates.



(a) (b)

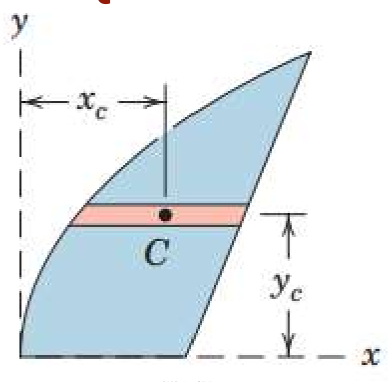
Centroids of Lines, Areas, and Volumes…

Choice of Element for Integration…

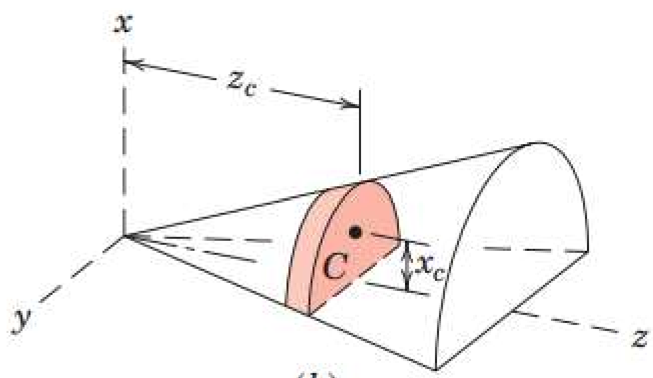
5. Centroidal Coordinate of Element.

When a first- or second order differential element is chosen, it is essential to use the coordinate of the centroid of the element for the moment arm in expressing the moment of the differential element. Thus, for the horizontal strip of area in Fig. (a), the moment of dA about the y-axis is $x_c dA$, where x_c is the x-coordinate of the centroid C of the element. Note that x_c is not the x which describes either boundary of the area. In the y-direction for this element the moment arm y_c of the centroid of the element is the same, in the limit, as the y-coordinates of the two boundaries.

As a second example, consider the solid half-cone of Fig. (b) with the semicircular slice of differential thickness as the element of volume. The moment arm for the element in the x-direction is the distance x_c to the centroid of the face of the element and not the x-distance to the boundary of the element. On the other hand, in the z-direction the moment arm z_c of the centroid of the element is the same as the z-coordinate of the element.



(a)



(b)

Centroids of Lines, Areas, and Volumes…

Choice of Element for Integration…

5. Centroidal Coordinate of Element…

With the concepts above, the equation (2) &(3) above can be rewritten as

$$\bar{x} = \frac{\int x_c \, dA}{A} \qquad \bar{y} = \frac{\int y_c \, dA}{A} \qquad \bar{z} = \frac{\int z_c \, dA}{A}$$

$$\bar{x} = \frac{\int x_c \, dV}{V} \qquad \bar{y} = \frac{\int y_c \, dV}{V} \qquad \bar{z} = \frac{\int z_c \, dV}{V}$$

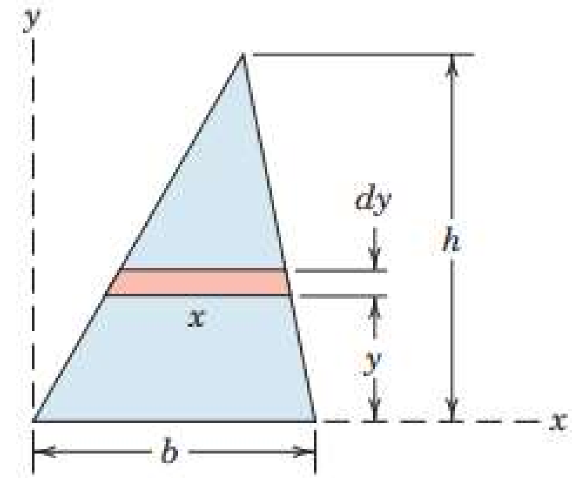
It is essential to recognize that the subscript c serves as a reminder that the moment arms appearing in the numerators of the integral expressions for moments are always the coordinates of the centroids of the particular elements chosen.

Keep in mind the equivalence between the moment of the resultant weight W and the sum (integral) of the moments of the elemental weights dW, to avoid mistakes in setting up the necessary mathematics. Recognition of the principle of moments will help in obtaining the correct expression for the moment arm x_c, y_c, or z_c of the centroid of the chosen differential element.

Centroids of Lines, Areas, and Volumes…

Example 1

Determine the distance $\bar{h}$ from the base of a triangle of altitude h to the centroid of its area.



Solution

- **Rectangular coordinate system is best**
- **The x-axis is taken to coincide with the base.**
- **Horizontal strip is best. Thus for this strip, we save one integration here by using the first-order element of area. Recognize that dA must be expressed in terms of the integration variable y; hence, x = ƒ(y) is required.**

B/c dA = x dy . By similar triangles x/(h - y) = b/h.
x= b(h-y)/h. Therefore dA= [b(h-y)/h]dy

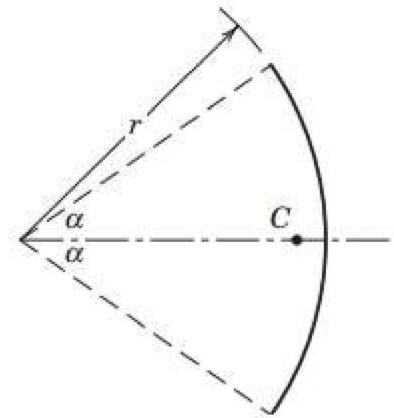
$$\left[A\bar{y} = \int y_c \, dA\right] \qquad \frac{bh}{2}\bar{y} = \int_0^h y \frac{b(h-y)}{h} dy = \frac{bh^2}{6} \qquad \bar{y} = \frac{h}{3}$$

Therefore $\bar{h}$=h/3 as it is measured from base and
$\bar{h}$=2h/3 as it is measured from apex

Centroids of Lines, Areas, and Volumes…

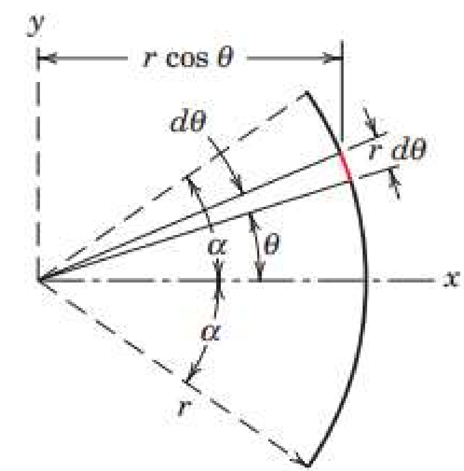
Examples 2

Locate the centroid of a circular arc as shown in the figure



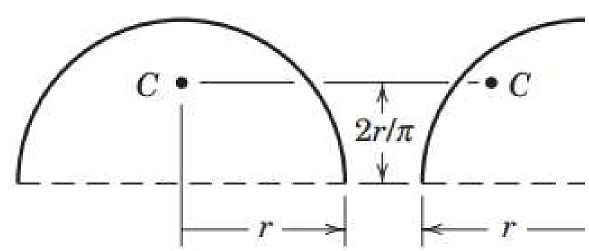
Solution: polar coordinates are preferable to rectangular coordinates to express the length of a circular arc.
Choosing the axis of symmetry as the x-axis makes $\bar{y}$=0. A differential element of arc has the length dL = rdθ expressed in polar coordinates, and the x-coordinate of the element is r cos θ .

- **Applying principle of moment** $[L\bar{x} = \int x\, dL]$
- **substituting L= 2*αr***



$$(2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r\cos\theta)\, r\, d\theta \qquad 2\alpha r\bar{x} = 2r^2 \sin\alpha \qquad \bar{x} = \frac{r\sin\alpha}{\alpha}$$

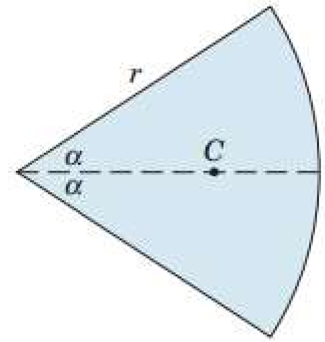
For a semicircular arc 2α=π, which gives x= 2r/π. By symmetry we see immediately that this result also applies to the quarter-circular arc when the measurement is made as shown



Centroids of Lines, Areas, and Volumes…

Examples 3

Locate the centroid of the area of a circular sector with respect to its vertex.



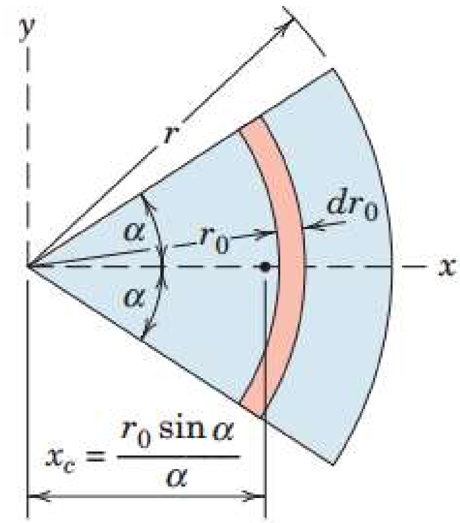
Solution I

The x-axis is chosen as the axis of symmetry, and is therefore automatically zero. We may cover the area by moving an element in the form of a partial circular ring, as shown in the figure, from the center to the outer periphery.

The radius of the ring is r_0 and its thickness is dr_0, so that its area is

$dA = 2r_0\alpha \, dr_0$

The x-coordinate to the centroid of the element from example 2 above is $x_c = r_0 \sin\alpha / \alpha$, where r_0 replaces r in the formula. Thus,



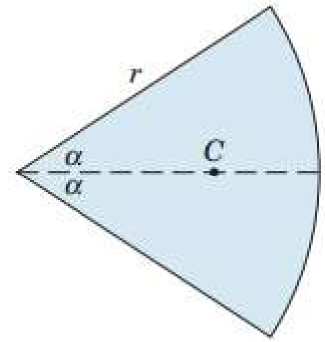
$$\left[A\bar{x} = \int x_c \, dA\right] \qquad \frac{2\alpha}{2\pi}(\pi r^2)\bar{x} = \int_0^r \left(\frac{r_0 \sin\alpha}{\alpha}\right)(2r_0\alpha \, dr_0)$$

$$r^2\alpha\bar{x} = \frac{2}{3}r^3 \sin\alpha \qquad \bar{x} = \frac{2}{3}\frac{r \sin\alpha}{\alpha}$$

Centroids of Lines, Areas, and Volumes…

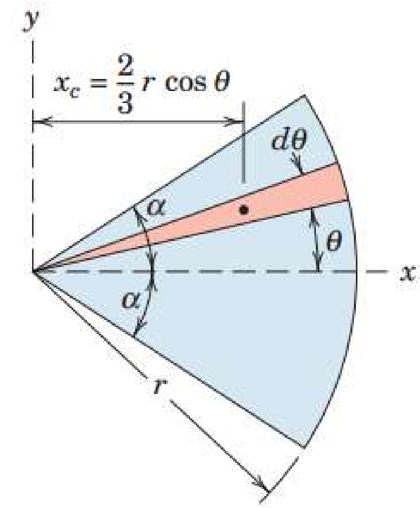
Examples 3…

Locate the centroid of the area of a circular sector with respect to its vertex.



Solution II

The area may also be covered by swinging a triangle of differential area about the vertex and through the total angle of the sector. This triangle, shown in the illustration, has an area dA=(r/2)*(r dθ), where higher-order terms are neglected. From example 2bthe centroid of the triangular element of area is two-thirds of its altitude from its vertex, so that the x-coordinate to the centroid of the element is xc=(2/3)*rcos θ. Applying

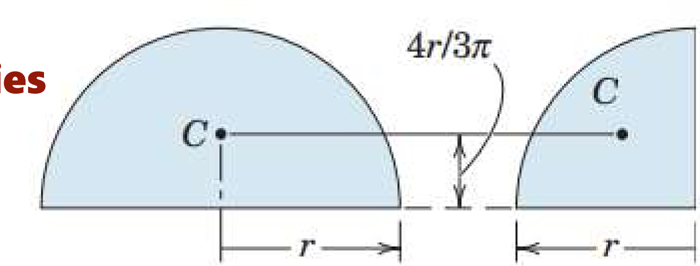


$$[A\bar{x} = \int x_c \, dA]$$

$$(r^2\alpha)\bar{x} = \int_{-\alpha}^{\alpha} \left(\tfrac{2}{3} r \cos\theta\right)\left(\tfrac{1}{2} r^2 \, d\theta\right)$$

$$r^2\alpha\bar{x} = \tfrac{2}{3} r^3 \sin\alpha \qquad \bar{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha}$$

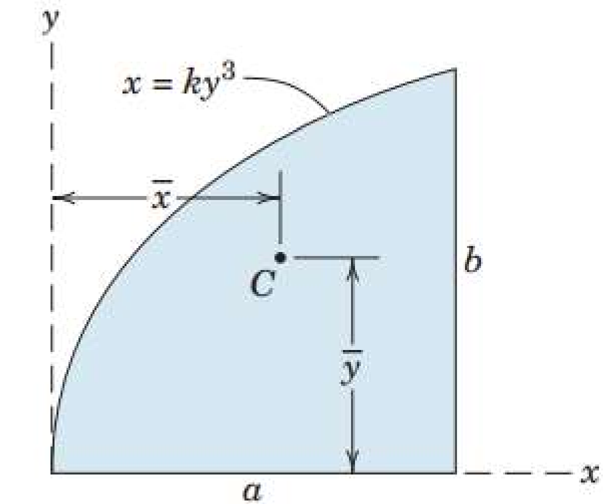
For a semicircular area 2α=π, which gives $\bar{x}$=4r/3π. By symmetry we see immediately that this result also applies to the quarter-circular area where the measurement is made as shown



Centroids of Lines, Areas, and Volumes…

Examples 4

Locate the centroid of the area under the curve $x = ky^3$ from $x = 0$ to $x = a$.

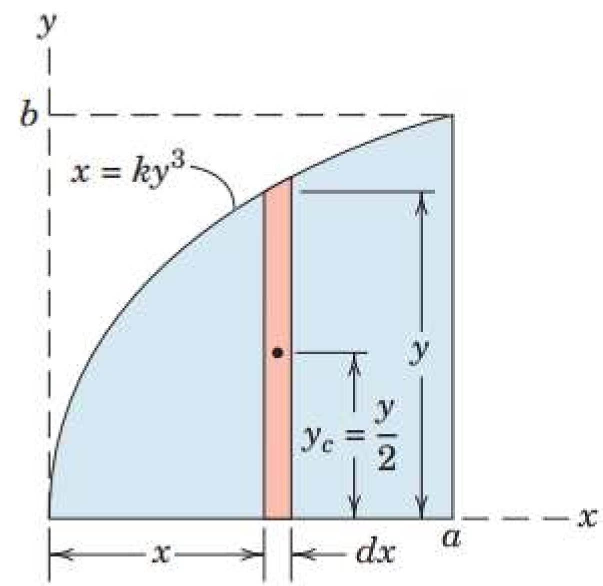


Solution I. A vertical element of area dA = y dx is chosen as shown in the figure. The x-coordinate of the centroid is found from

$$\left[A\bar{x} = \int x_c \, dA\right] \qquad \bar{x}\int_0^a y\,dx = \int_0^a xy\,dx$$

Substituting $y = (x/k)^{1/3}$ and $k = a/b^3$ and integrating give

$$\frac{3ab}{4}\bar{x} = \frac{3a^2b}{7} \qquad \bar{x} = \frac{4}{7}a$$



To calculate $\bar{y}$, the y-coordinate to the centroid of the rectangular strip is $y_c = y/2$.

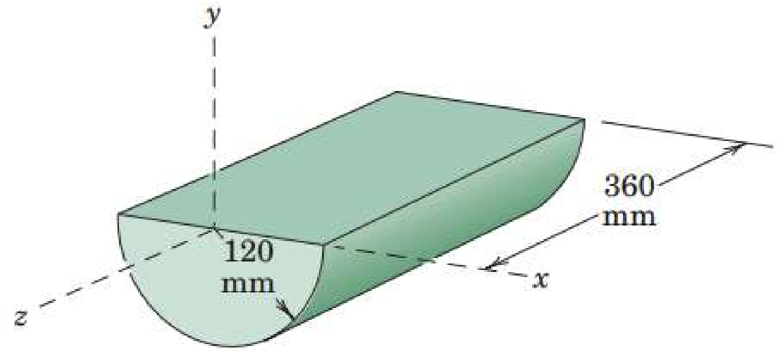
$$\left[A\bar{y} = \int y_c \, dA\right] \qquad \frac{3ab}{4}\bar{y} = \int_0^a \left(\frac{y}{2}\right) y\,dx$$

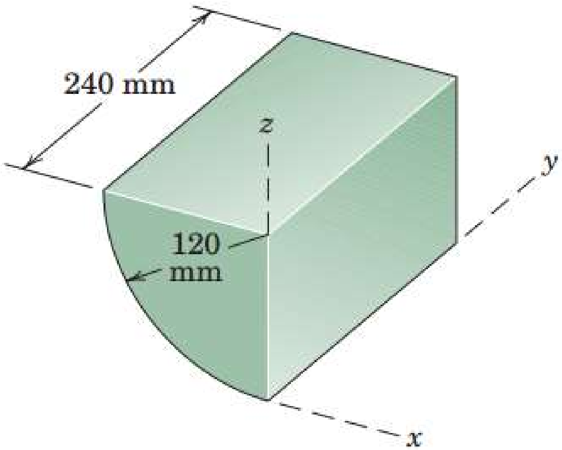
Substituting $y = b(x/a)^{1/3}$ and integrating give

$$\frac{3ab}{4}\bar{y} = \frac{3ab^2}{10} \qquad \bar{y} = \frac{2}{5}b$$

Exercise

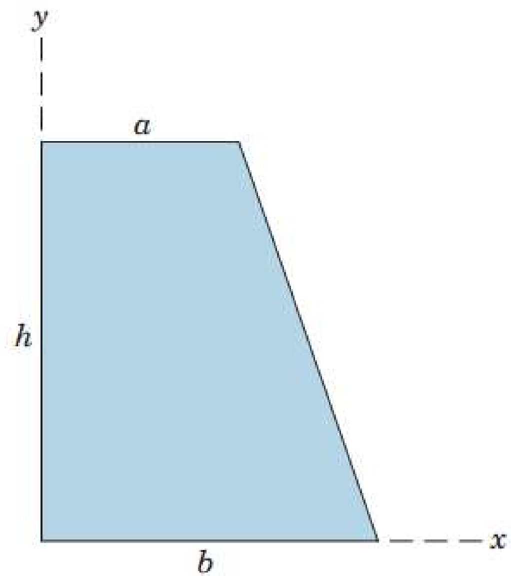
1. Specify the x-, y-, and z-coordinates of the mass center of the homogeneous semicylinder





2. Specify the x-, y-, and z-coordinates of the mass center of the quadrant of the homogeneous solid cylinder.

3) Determine the x- and y-coordinates of the centroid of the trapezoidal area.



Centroid of Composite Bodies and Figures

When a body or figure can be conveniently divided into several parts whose mass centers are easily determined, we use the principle of moments and treat each part as a finite element of the whole.

If Its parts have masses m_1, m_2, m_3 with the respective mass-center coordinates x_1, x_2, x_3 in the x-direction.
The moment principle gives $(m_1 + m_2 + m_3)\bar{X} = m_1\bar{x}_1 + m_2\bar{x}_2 + m_3\bar{x}_3$

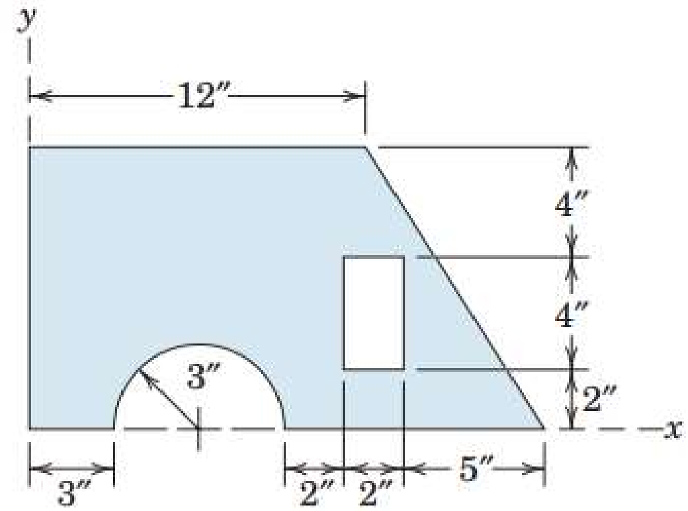
Where $\bar{X}$ is the x-coordinate of the center of mass of the whole.
Similar relations hold for the other two coordinate directions.

$$\bar{X} = \frac{\Sigma m\bar{x}}{\Sigma m} \qquad \bar{Y} = \frac{\Sigma m\bar{y}}{\Sigma m} \qquad \bar{Z} = \frac{\Sigma m\bar{z}}{\Sigma m}$$

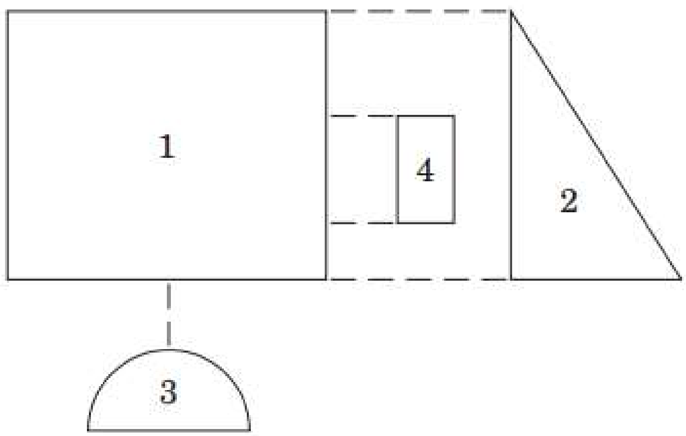
Analogous relations hold for composite lines, areas, and volumes, where the m′s are replaced by L′s, A′s, and V′s, respectively. Note that if a hole or cavity is considered one of the component parts of a composite body or figure, the corresponding mass represented by the cavity or hole is treated as a negative quantity.

Example 1

1)Locate the centroid of the shaded area



***Solution**: The composite area is divided into the four elementary shapes shown in the Figure below*



Note that the areas of the "holes" (parts 3 and 4) are taken as negative in the following table:

PART	A in.^{2}	$\bar{x}$ in.	$\bar{y}$ in.	$\bar{x}A$ in.^{3}	$\bar{y}A$ in.^{3}
1	120	6	5	720	600
2	30	14	10/3	420	100
3	−14.14	6	1.273	−84.8	−18
4	−8	12	4	−96	−32
TOTALS	127.9			959	650

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right]$$

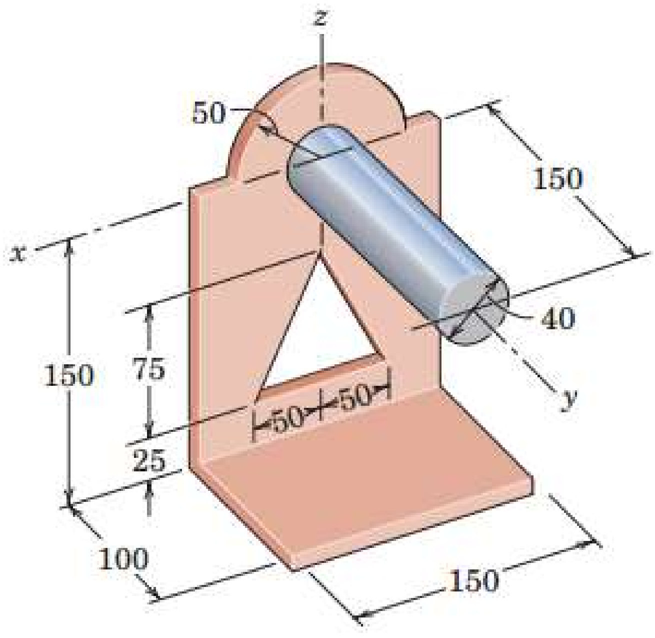
$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right]$$

$$\bar{X} = \frac{959}{127.9} = 7.50 \text{ in.}$$

$$\bar{Y} = \frac{650}{127.9} = 5.08 \text{ in.}$$

Example 2

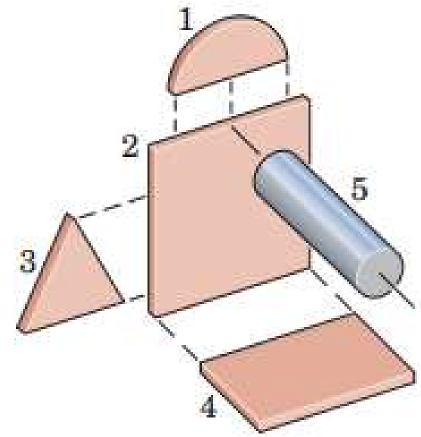
Locate the center of mass of the bracket-and-shaft combination. The vertical face is made from sheet metal which has a mass of 25 kg/m^2. The material of the horizontal base has a mass of 40 kg/m^2, and the steel shaft has a density of 7.83 Mg/m^3.



Dimensions in millimeters

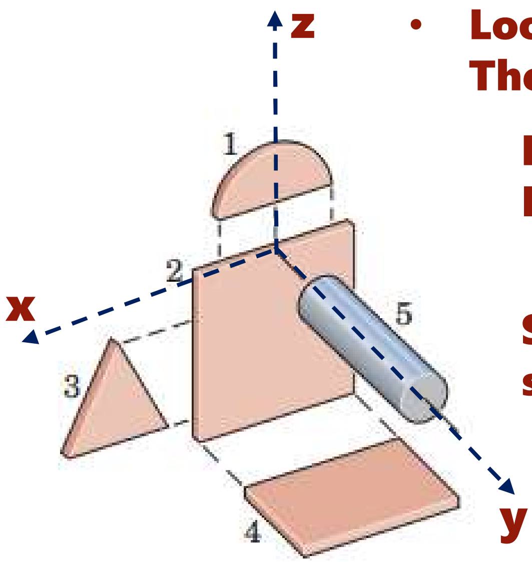
Solution:

The composite body may be considered to be composed of the five elements shown in the lower portion of the illustration. The triangular part will be taken as a negative mass. For the reference axes indicated it is clear by symmetry that the x-coordinate of the center of mass is zero. The mass m of each part is easily calculated



Example 2…

Solution…



- **Look at the rectangular coordinate system chosen. Then calculate mass and centroid for each parts**

For part 1, m=(25kg/m²)*3.14*0.05²/2=0.098 kg

Its centroid are $\overline{x}=\overline{y}=0$, and

$$\bar{z} = \frac{4r}{3\pi} = \frac{4(50)}{3\pi} = 21.2 \text{ mm}$$

Similarly calculated for other parts and summarized in the table shown below

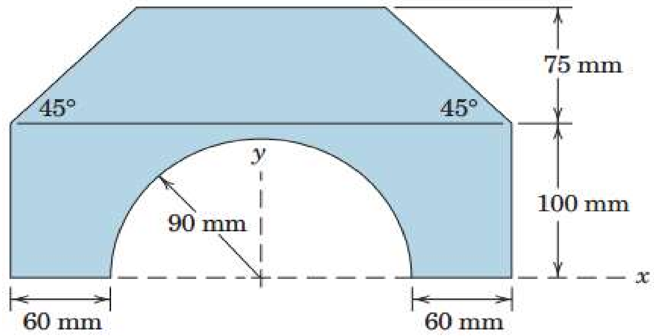
PART	m kg	$\bar{y}$ mm	$\bar{z}$ mm	$m\bar{y}$ kg·m	$m\bar{z}$ kg·mm
1	0.098	0	21.2	0	2.08
2	0.562	0	−75.0	0	−42.19
3	−0.094	0	−100.0	0	9.38
4	0.600	50.0	−150.0	30.0	−90.00
5	1.476	75.0	0	110.7	0
TOTALS	2.642			140.7	−120.73

$$\left[\overline{Y} = \frac{\Sigma m\bar{y}}{\Sigma m}\right] \qquad \overline{Y} = \frac{140.7}{2.642} = 53.3 \text{ mm}$$

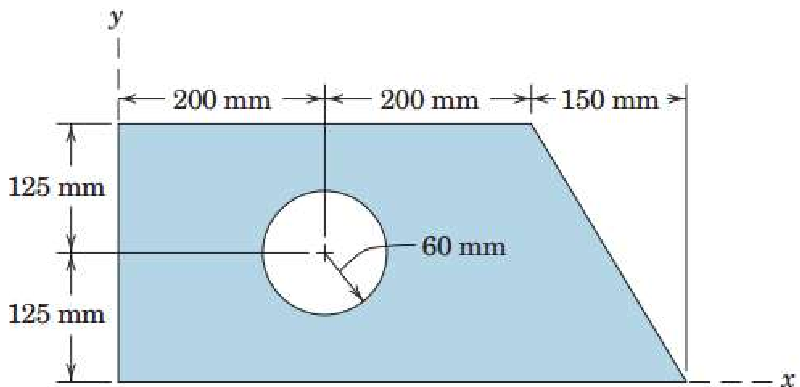
$$\left[\overline{Z} = \frac{\Sigma m\bar{z}}{\Sigma m}\right] \qquad \overline{Z} = \frac{-120.73}{2.642} = -45.7 \text{ mm}$$

Exercise

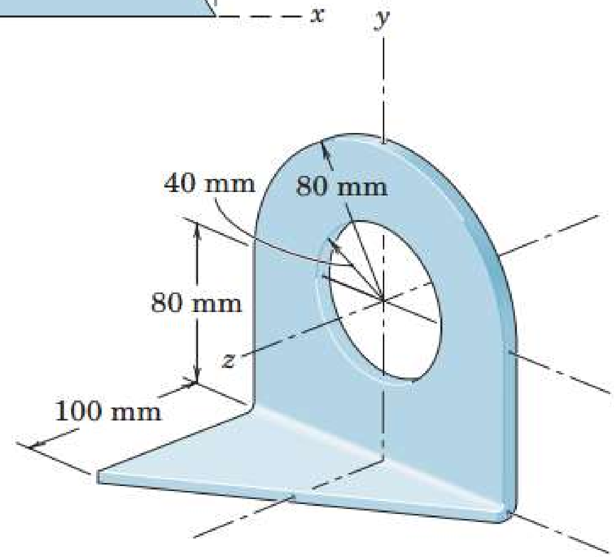
1)Determine the y-coordinate of the centroid of the shaded are



2) Determine the coordinates of the centroid of the shaded area

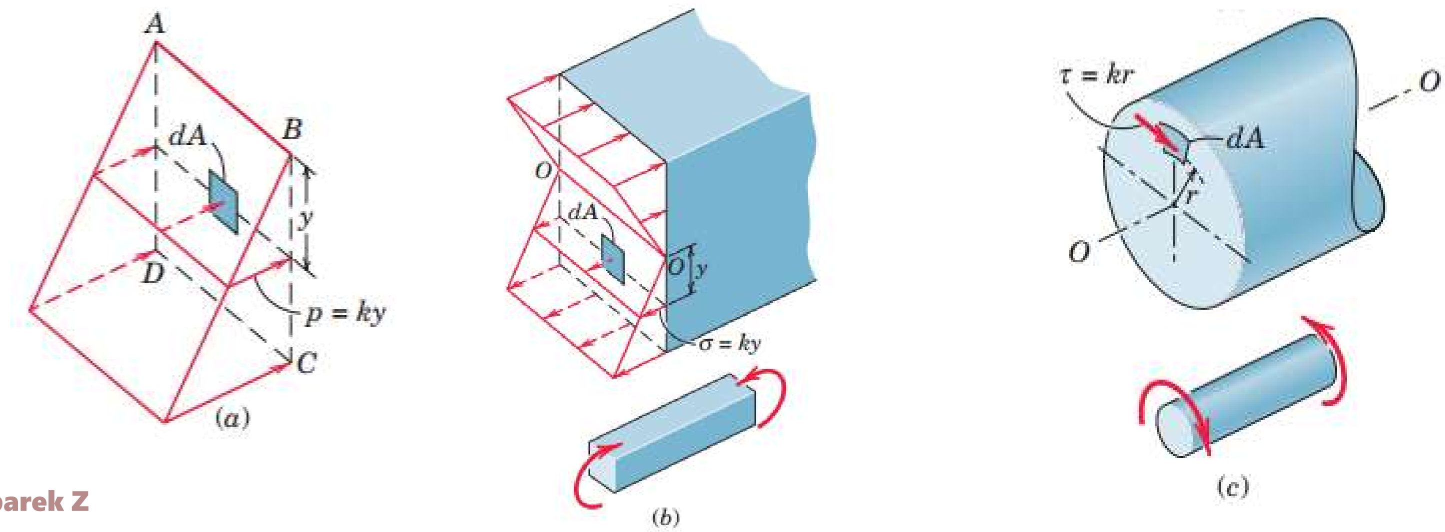


3) Determine the coordinates of the center of mass of the bracket, which is made from a plate of uniform thickness.



Introduction

- When forces are distributed continuously over an area on which they act, it is often necessary to calculate the moment of these forces about some axis either in or perpendicular to the plane of the area. The intensity of the force (pressure or stress) is proportional to the distance of the force from the moment axis.
- The elemental force acting on an element of area, then, is proportional to distance times differential area, and the elemental moment is proportional to distance squared times differential area. We see, therefore that the total moment involves an integral that has the form $\int (\text{distance})^2 \, d(\text{area})$. This integral is known as the moment of inertia or the second moment of the area.



Introduction

- **In Figure (a) above the surface area ABCD is subjected to a distributed pressure p whose intensity is proportional to the distance y from the axis AB. The moment about AB due to the pressure on the element of area dA is py*dA =ky^2*dA. Thus, the integral in question appears when the total moment M is evaluated as**

$$M = k \int y^2 \, dA$$

- **Figure (b) shown the distribution of stress acting on a transverse section of a simple elastic beam bent by equal and opposite couples applied to its ends. At any section of the beam, a linear distribution of force intensity or stress σ , given by σ= ky, is present. The stress is positive (tensile) below the axis O–O and negative (compressive) above the axis. We see that the elemental moment about the axis O–O is dM =y(dA) =ky^{2}dA. Thus, the same integral appears when the total moment** $M = k \int y^2 \, dA$ **is to be evaluated**

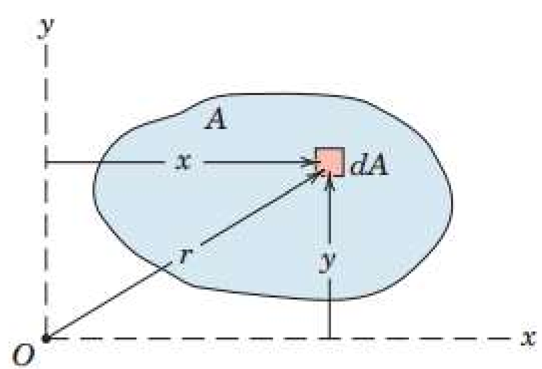
- ***In Figure (c) which shows a circular shaft subjected to a twist or torsional moment. Within the elastic limit of the material, this moment is resisted at each cross section of the shaft by a distribution of tangential or shear stress τ, which is proportional to the radial distance r from the center. Thus, τ= kr, and the total moment about the central axis is*** $M = \int r(\tau \, dA) = k \int r^2 \, dA$

Here the integral differs from that in the preceding cases in that the area is normal instead of parallel to the moment axis and in that r is a radial coordinate instead of a rectangular one.

Rectangular and Polar Moments of Inertia

The integral illustrated in the preceding examples is generally called the moment of inertia of the area about the axis in question, a more fitting term is the second moment of area, since the first moment ydA is multiplied by the moment arm y to obtain the second moment for the element dA. The word inertia appears in the terminology by reason of the similarity between the mathematical form of the integrals for second moments of areas and those for the resultant moments of the so called inertia forces in the case of rotating bodies.

Consider the area A in the x-y plane, Figure. The moments of inertia of the element dA about the x- and y-axes are, by definition, $dI_x = y^2 dA$ and $dI_y = x^2 dA$, respectively. The moments of inertia of A about the same axes are therefore



$$I_x = \int y^2 \, dA$$

$$I_y = \int x^2 \, dA$$

These expressions are called rectangular moments of inertia

Rectangular and Polar Moments of Inertia…

The moment of inertia of dA about the pole O (z-axis) is, by similar definition, $dI_z = r^2\, dA$. The moment of inertia of the entire area about O is

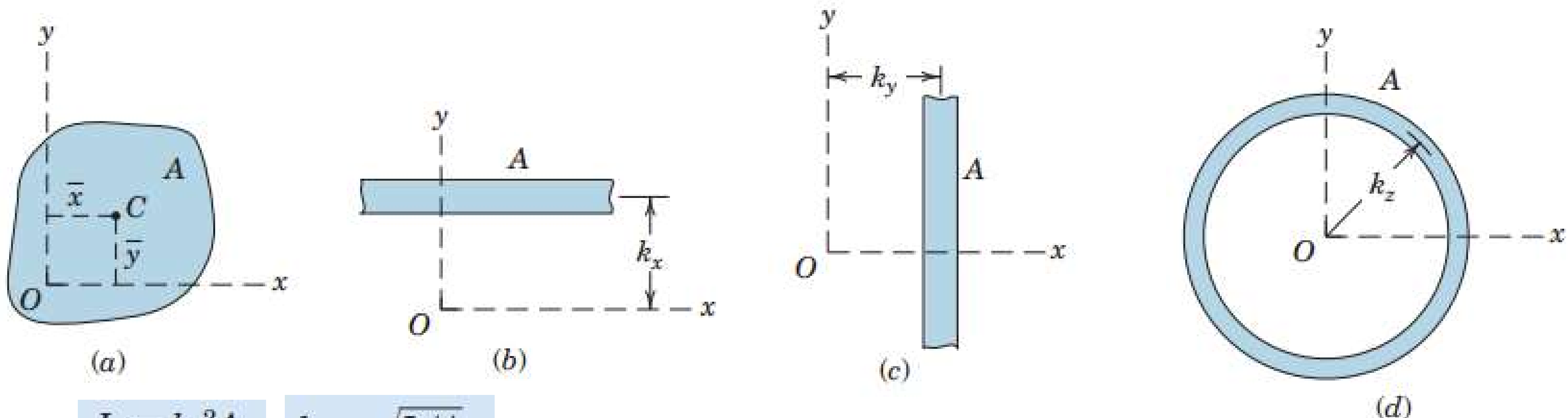
$$I_z = \int r^2\, dA$$

This expression is called the polar moment of inertia.

Because $x^2 + y^2 = r^2$, it is clear that

$$I_z = I_x + I_y$$

Radius of Gyration



$$I_x = k_x^2 A \qquad k_x = \sqrt{I_x/A}$$

$$I_y = k_y^2 A \qquad k_y = \sqrt{I_y/A}$$

$$I_z = k_z^2 A \qquad k_z = \sqrt{I_z/A}$$

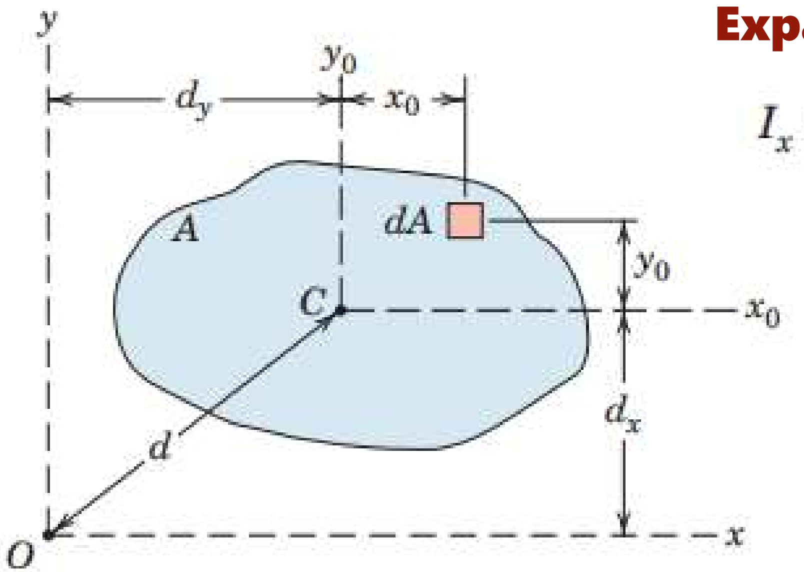
$$k_z^2 = k_x^2 + k_y^2$$

The radius of gyration, then, is a measure of the distribution of the area from the axis in question

Transfer of Axes(Parallel axis Theorem)

- ✓ **The moment of inertia of an area about a noncentroidal axis may be easily expressed in terms of the moment of inertia about a parallel centroidal axis. In Figure below the x_0-y_0 axes pass through the centroid C of the area. Let us now determine the moments of inertia of the area about the parallel x-y axes. By definition, the moment of inertia of the element dA about the x-axis is**

$$dI_x = (y_0 + d_x)^2\, dA$$



Expanding and integrating give us

$$I_x = \int y_0^{\,2}\, dA + 2d_x \int y_0\, dA + d_x^{\,2} \int dA$$

- **We see that the first integral is by definition the moment of inertia $\bar{I}_x$ about the centroidal x_0-axis. The second integral is zero, since**

$$\int y_0\, dA = A\bar{y}_0$$

$\bar{y}_0$ is automatically zero with the centroid on the x_0-axis. The third term is simply Ad_x^2

Transfer of Axes…

Thus, the expression for I_x and the similar expression for I_y become

$$I_x = \bar{I}_x + Ad_x^{\,2}$$
$$I_y = \bar{I}_y + Ad_y^{\,2}$$

the sum of these two equations gives $I_z = \bar{I}_z + Ad^2$

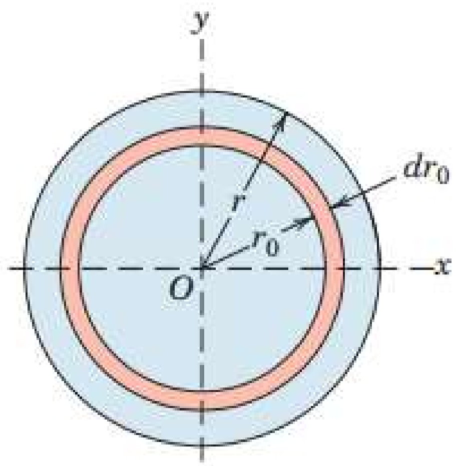
These above three equations are the so-called parallel-axis theorems.

- ✓ **Two points in particular should be noted. First, the axes between which the transfer is made must be parallel, and second, one of the axes must pass through the centroid of the area.**
- ✓ **If a transfer is desired between two parallel axes neither of which passes through the centroid, it is first necessary to transfer from one axis to the parallel centroidal axis and then to transfer from the centroidal axis to the second axis.**
- ✓ **The parallel-axis theorems also hold for radii of gyration. With substitution of the definition of k into equations above, the transfer relation becomes** $k^2 = \bar{k}^2 + d^2$

Where $\bar{k}$ is the radius of gyration about a centroidal axis parallel to the axis about which k applies and d is the distance between the two axes.

Example 1

Calculate the moments of inertia of the area of a circle about a diametric axis and about the polar axis through the center. Specify the radii of gyration.



Solution:-

A differential element of area in the form of a circular ring may be used for the calculation of the moment of inertia about the polar z-axis through O since all elements of the ring are equidistant from O. The elemental area is

$dA = 2\pi r_0\, dr_0$, and thus,

$$[I_z = \int r^2\, dA] \qquad I_z = \int_0^r r_0^{\,2}(2\pi r_0\, dr_0) = \frac{\pi r^4}{2} = \frac{1}{2}Ar^2$$

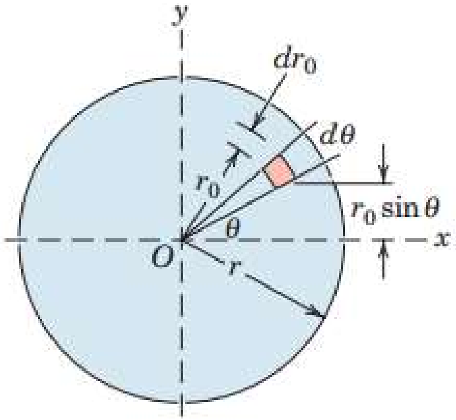
The polar radius of gyration is $\left[k = \sqrt{\frac{I}{A}}\right]$ $\quad k_z = \frac{r}{\sqrt{2}}$

By symmetry $I_x = I_y$; using $[I_z = I_x + I_y]$ **then** $I_x = \frac{1}{2}I_z = \frac{\pi r^4}{4} = \frac{1}{4}Ar^2 = \frac{\pi r^4}{4}$

The radius of gyration about the diametric axis is $\left[k = \sqrt{\frac{I}{A}}\right]$ $\quad k_x = \frac{r}{2}$

Example 1

Solution:- Alternatively using the differential element as shown in the Figure below. Then $dA = r_0 \, dr_0 \, d\theta$



By definition

$$\left[I_x = \int y^2 \, dA\right]$$

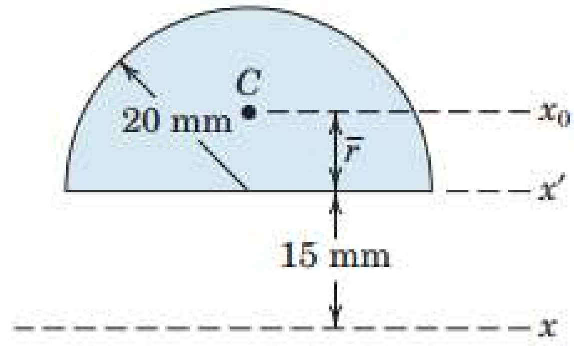
$$I_x = \int_0^{2\pi} \int_0^{r} (r_0 \sin\theta)^2 r_0 \, dr_0 \, d\theta = \int_0^{2\pi} \frac{r^4 \sin^2\theta}{4} \, d\theta$$

$$= \frac{r^4}{4} \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{\pi r^4}{4}$$

The radius of gyration about the diametric x- axis is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_x = \frac{r}{2}$$

Example 3



Find the moment of inertia about the x-axis of the semicircular area.

Solution: The moment of inertia of the semicircular area about the *x*`-axis is one-half of that for a complete circle about the same axis. Thus, from the results of example 1

$$I_{x'} = \frac{1}{2}\frac{\pi r^4}{4} = \frac{20^4\pi}{8} = 2\pi(10^4)\ \text{mm}^4$$

We obtain the moment of inertia $\bar{I}_x$ about the parallel centroidal axis x_0 next. Transfer is made through the distance

$$\bar{r} = 4r/(3\pi) = (4)(20)/(3\pi) = 80/(3\pi)\ \text{mm}$$

by the parallel-axis theorem. Hence, $[\bar{I} = I - Ad^2]$

$$\bar{I} = 2(10^4)\pi - \left(\frac{20^2\pi}{2}\right)\left(\frac{80}{3\pi}\right)^2 = 1.755(10^4)\ \text{mm}^4$$

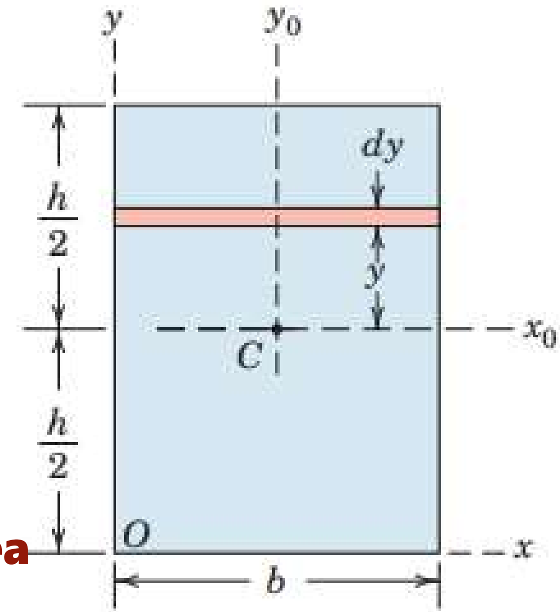
Finally, we transfer from the centroidal x_0-axis to the x-axis. Thus,

$[I = \bar{I} + Ad^2]$

$$I_x = 1.755(10^4) + \left(\frac{20^2\pi}{2}\right)\left(15 + \frac{80}{3\pi}\right)^2 = 1.755(10^4) + 34.7(10^4) = 36.4(10^4)\ \text{mm}^4$$

Example 3

Determine the moments of inertia of the rectangular area about the centroidal x_0- and y_0-axes, the centroidal polar axis z_0 through C, the x-axis, and the polar axis z through O.



Solution:- For the calculation of the moment of inertia $\bar{I}_x$ about the x_0-axis, a horizontal strip of area $dA=b*d_y$ is chosen so that all elements of the strip have the same y-coordinate. Thus,

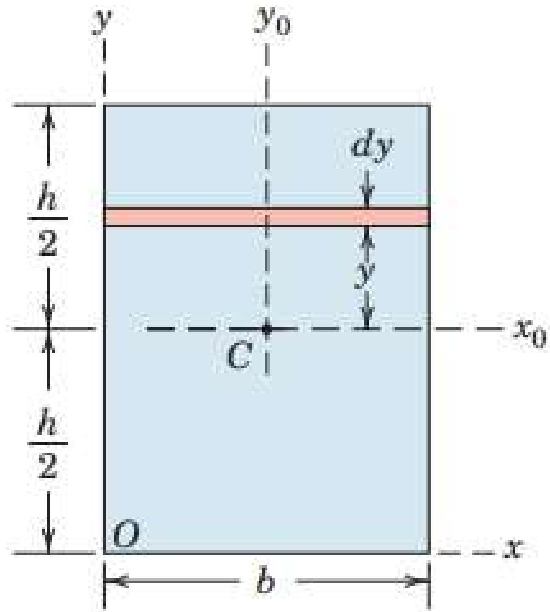
$$\left[I_x = \int y^2\,dA\right] \quad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\,dy = \frac{1}{12}bh^3$$

By interchange of symbols, the moment of inertia about the centroidal y_0-axis is $\bar{I}_y = \frac{1}{12}hb^3$

The centroidal polar moment of inertia is $\left[\bar{I}_z = \bar{I}_x + \bar{I}_y\right]$

$$\bar{I}_z = \frac{1}{12}(bh^3 + hb^3) = \frac{1}{12}A(b^2 + h^2)$$

Example 3...



Solution: continued...

By the parallel-axis theorem the moment of inertia about the x-axis is

$$[I_x = \bar{I}_x + Ad_x^{\,2}]$$

$$I_x = \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3 = \frac{1}{3}Ah^2$$

We also obtain the polar moment of inertia about O by the parallel-axis theorem, which gives us

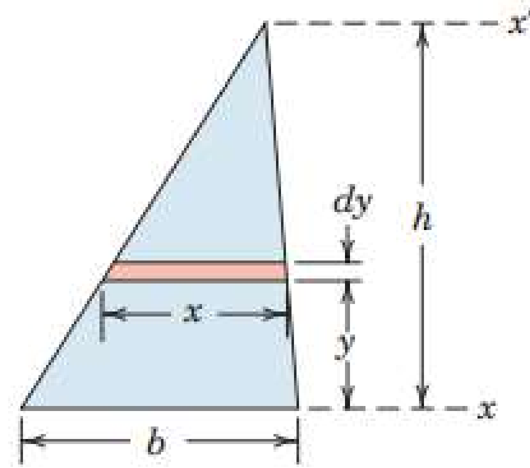
$$[I_z = \bar{I}_z + Ad^2]$$

$$I_z = \frac{1}{12}A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

$$I_z = \frac{1}{3}A(b^2 + h^2)$$

Example 4

Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.



Solution:

A strip of area parallel to the base is selected as shown in the figure, and it has the area

$$dA = x\,dy = [(h - y)b/h]\,dy.$$

By definition $\left[I_x = \int y^2\,dA\right]$

$$I_x = \int_0^h y^2 \frac{h-y}{h} b\,dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12}$$

By the parallel-axis theorem the moment of inertia I_x about an axis through the centroid, a distance h/3 above the x-axis, is

$[\bar{I} = I - Ad^2]$

$$\bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36}$$

A transfer from the centroidal axis to the x`-axis through the vertex gives

$[I = \bar{I} + Ad^2]$

$$I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4}$$

Moment of inertia of Composite Areas

It is frequently necessary to calculate the moment of inertia of an area composed of a number of distinct parts of simple and calculable geometric shape. Because a moment of inertia is the integral or sum of the products of distance squared times element of area, it follows that the moment of inertia of a positive area is always a positive quantity. The moment of inertia of a composite area about a particular axis is therefore simply the sum of the moments of inertia of its component parts about the same axis. It is often convenient to regard a composite area as being composed of positive and negative parts. We may then treat the moment of inertia of a negative area as a negative quantity

For such an area in the x-y plane, for example, and with the notation of Fig. A/4, where is the same as and is the same as the tabulation would include

Part	Area, A	d_x	d_y	Ad_x^2	Ad_y^2	$\bar{I}_x$	$\bar{I}_y$
Sums	ΣA			ΣAd_x^2	ΣAd_y^2	$\Sigma\bar{I}_x$	$\Sigma\bar{I}_y$

From the sums of the four columns, then, the moments of inertia for the composite area about the x- and y-axes become

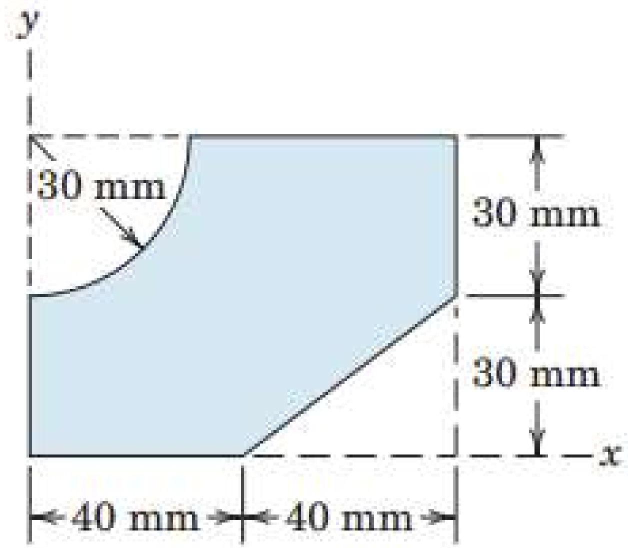
$$I_x = \Sigma\bar{I}_x + \Sigma Ad_x^2$$

$$I_y = \Sigma\bar{I}_y + \Sigma Ad_y^2$$

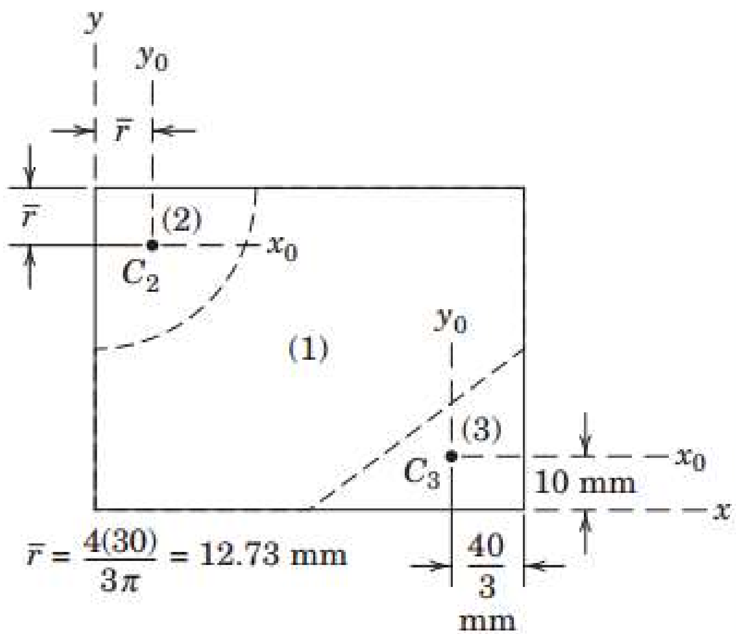
Moment of inertia of Composite Areas…

Example

Determine the moments of inertia about the x- and y-axes for the shaded area. Also determine the radius of gyration about x-axis



Solution: **The given area is subdivided into the three subareas shown—a rectangular (1), a quarter-circular (2), and a triangular (3) area. Two of the subareas are "holes" with negative areas. Centroidal axes are shown for areas (2) and (3), and the locations of centroids and are from Table D/3.**



Moment of inertia of Composite Areas…

Example solution …continued

PART	A mm^2	d_x mm	d_y mm	Ad_x^2 mm^3	Ad_y^2 mm^3	$\bar{I}_x$ mm^4	$\bar{I}_y$ mm^4
1	$80(60)$	30	40	$4.32(10^6)$	$7.68(10^6)$	$\frac{1}{12}(80)(60)^3$	$\frac{1}{12}(60)(80)^3$
2	$-\frac{1}{4}\pi(30)^2$	$(60-12.73)$	12.73	$-1.579(10^6)$	$-0.1146(10^6)$	$-\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)30^4$	$-\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)30^4$
3	$-\frac{1}{2}(40)(30)$	$\frac{30}{3}$	$\left(80-\frac{40}{3}\right)$	$-0.06(10^6)$	$-2.67(10^6)$	$-\frac{1}{36}40(30)^3$	$-\frac{1}{36}(30)(40)^3$
TOTALS	3490			$2.68(10^6)$	$4.90(10^6)$	$1.366(10^6)$	$2.46(10^6)$

$[I_x = \Sigma \bar{I}_x + \Sigma A d_x^2]$ $\quad I_x = 1.366(10^6) + 2.68(10^6) = 4.05(10^6)\ \mathrm{mm}^4$ *Ans.*

$[I_y = \Sigma \bar{I}_y + \Sigma A d_y^2]$ $\quad I_y = 2.46(10^6) + 4.90(10^6) = 7.36(10^6)\ \mathrm{mm}^4$ *Ans.*

The net area of the figure is $A = 60(80) - \frac{1}{4}\pi(30)^2 - \frac{1}{2}(40)(30) = 3490\ \mathrm{mm}^2$ so that the radius of gyration about the x-axis is

$$k_x = \sqrt{I_x/A} = \sqrt{4.05(10^6)/3490} = 34.0\ \mathrm{mm}$$ *Ans.*

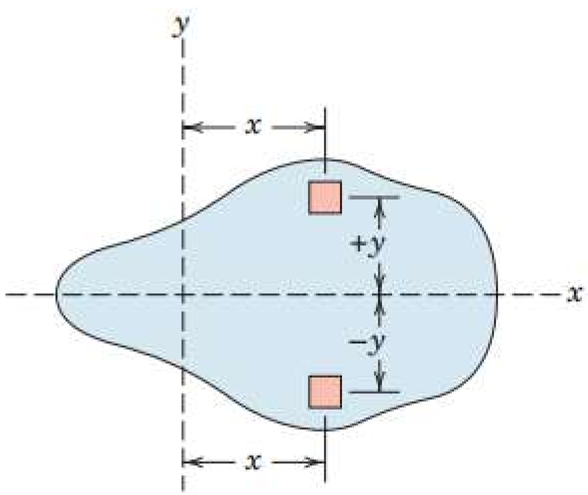
Products of Inertia and Rotation of Axes

In certain problems involving unsymmetrical cross sections and in the calculation of moments of inertia about rotated axes, an expression $dI_{xy} = xy\,dA$ occurs, which has the integrated form

$$I_{xy} = \int xy\,dA$$

where x and y are the coordinates of the element of area $dA = dx\,dy$.

The quantity I_{xy} is called the product of inertia of the area A with respect to the x-y axes. Unlike moments of inertia, which are always positive for positive areas, the product of inertia may be positive, negative, or zero.



The product of inertia is zero whenever either of the reference axes is an axis of symmetry, such as the x-axis for the area in Figure. Here we see that the sum of the terms x*(+y)*dA and x*(-y)*dA due to symmetrically placed elements vanishes. Because the entire area may be considered as composed of pairs of such elements, it follows that the product of inertia I_{xy} for the entire area is zero

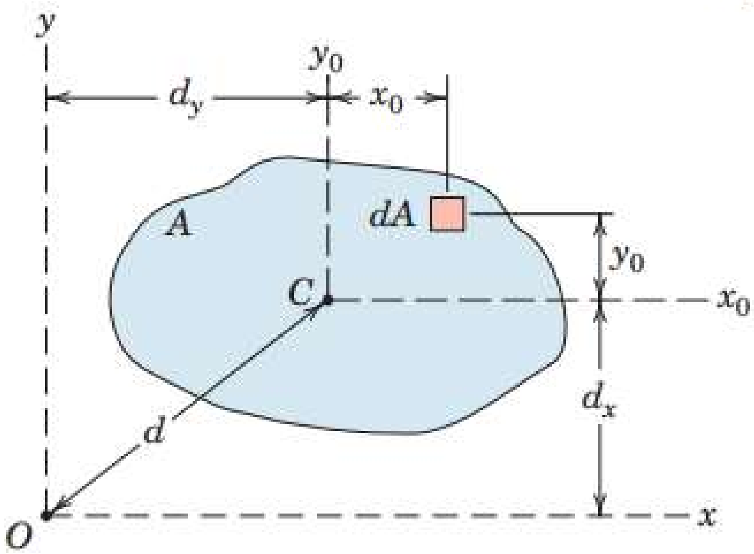
Products of Inertia and Rotation of Axes…

Transfer of Axes Theorem

By definition the product of inertia of the area A in Figure below with respect to the x- and y-axes in terms of the coordinates x_0, y_0 to the centroidal axes is

$$I_{xy} = \int (x_0 + d_y)(y_0 + d_x)\, dA$$

$$= \int x_0 y_0\, dA + d_x \int x_0\, dA + d_y \int y_0\, dA + d_x d_y \int dA$$



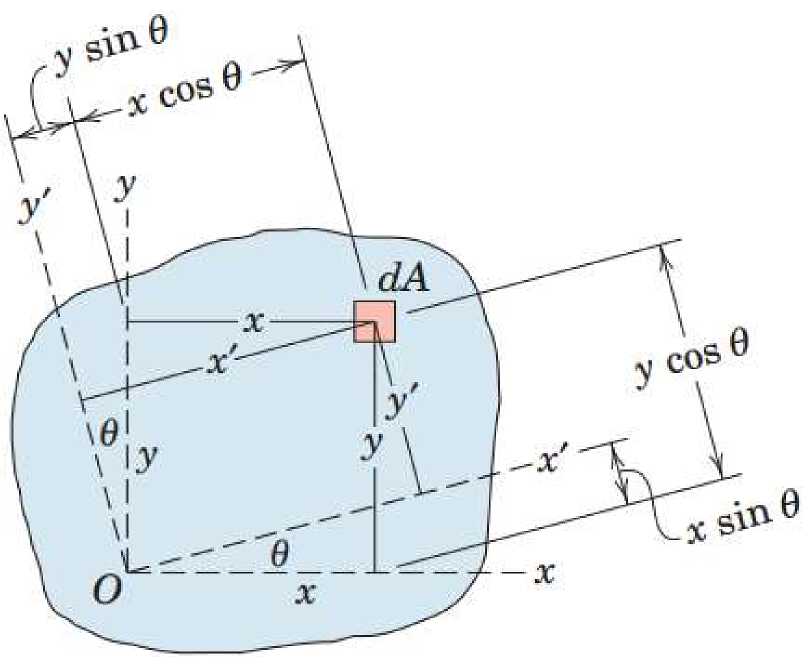
The first integral is by definition the product of inertia about the centroidal axes, which we write as $\bar{I}_{xy}$. The middle two integrals are both zero because the first moment of the area about its own centroid is necessarily zero. The fourth term is merely $d_x d_y A$.
Thus, the transfer-of axis theorem for products of inertia becomes

$$I_{xy} = \bar{I}_{xy} + d_x d_y A$$

Products of Inertia and Rotation of Axes…

Rotation of axes

The product of inertia is useful when we need to calculate the moment of inertia of an area about inclined axes. This consideration leads directly to the important problem of determining the axes about which the moment of inertia is a maximum and a minimum.



- **The moments of inertia of the area about the *x*`- and *y*`-axes are**

$$I_{x'} = \int y'^2\, dA = \int (y\cos\theta - x\sin\theta)^2\, dA$$

$$I_{y'} = \int x'^2\, dA = \int (y\sin\theta + x\cos\theta)^2\, dA$$

Expanding and substituting the trigonometric identities

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Products of Inertia and Rotation of Axes…

Rotation of axes…

…and the defining relations for I_x, I_y, I_{xy} give us

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

Equation (*)

In a similar manner we write the product of inertia about the inclined axes as

$$I_{x'y'} = \int x'y'\,dA = \int (y\sin\theta + x\cos\theta)(y\cos\theta - x\sin\theta)\,dA$$

Expanding and substituting the trigonometric identities

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta$$

and the defining relations for I_x, I_y, I_{xy} give us

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

Equation (**)

Products of Inertia and Rotation of Axes…

Rotation of axes…

Adding the equations

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

gives $I_{x`} + I_{y`} = I_x + I_y = I_z$, the polar moment of inertia about *O*

The angle which makes $I_{x`}$ and $I_{y`}$ either maximum or minimum may be determined by setting the derivative of either $I_{x`}$ or $I_{y`}$ with respect to θ equal to zero. Thus,

$$\frac{dI_{x'}}{d\theta} = (I_y - I_x)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

Denoting this critical angle by α gives

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}$$

Equation (***)

Equation (***) gives two values for 2α which differ by µ, since tan 2α =tan (2α+µ). Consequently the two solutions for α will differ by µ/2. One value defines the axis of maximum moment of inertia, and the other value defines the axis of minimum moment of inertia. These two rectangular axes are called the principal axes of inertia.

Products of Inertia and Rotation of Axes…

Rotation of axes…

When we substitute Equation(***) for the critical value of 2α in Equation(**), we see that the product of inertia is zero for the principal axes of inertia.
Substitution of sin 2α and cos 2α, obtained from Equation(***), for sin 2θ and cos 2θ in Equation(*) gives the expressions for the principal moments of inertia as

$$I_{\max} = \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}{}^2}$$

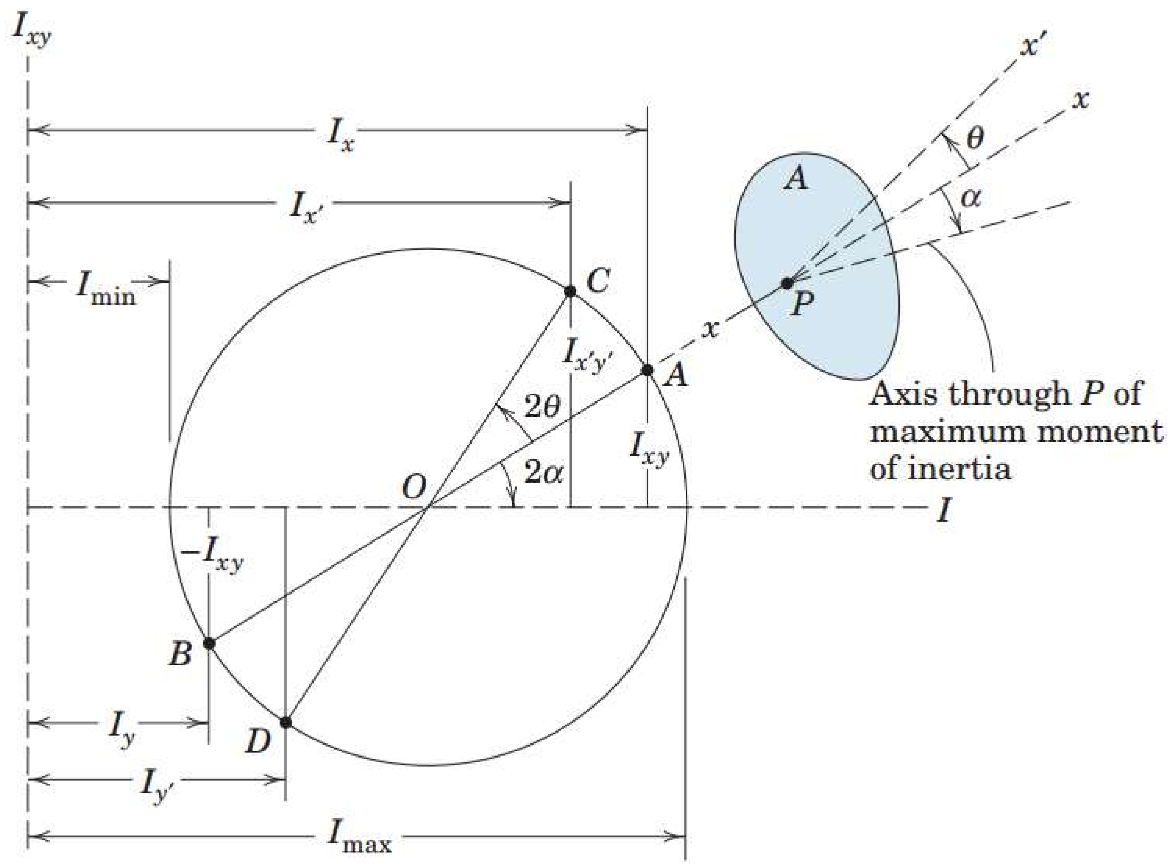
$$I_{\min} = \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}{}^2}$$

Products of Inertia and Rotation of Axes…

Mohr's Circle of Inertia

We may represent the relations in Eqs. A/9, A/9*a*, A/10, and A/11 graphically by a diagram called *Mohr's circle*. For given values of I_x, I_y, and I_{xy} the corresponding values of and may be determined from the diagram for any desired angle θ.

A horizontal axis for the measurement of moments of inertia and a vertical axis for the measurement of products of inertia are first selected, see Figure. Next, point A, which has the coordinates (I_x, I_{xy}), and point B, which has the coordinates (I_y, $-I_{xy}$), are located.



We now draw a circle with these two points as the extremities of a diameter. The angle from the radius OA to the horizontal axis is 2α or twice the angle from the x-axis of the area in question to the axis of maximum moment of inertia.

Products of Inertia and Rotation of Axes…

Mohr's Circle of Inertia

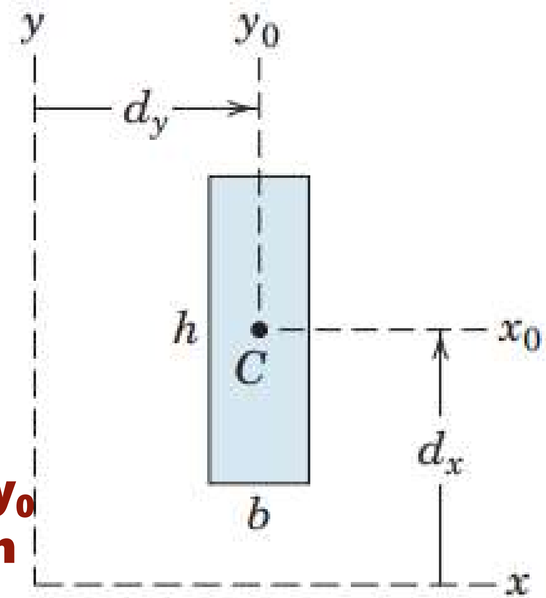
We now draw a circle with these two points as the extremities of a diameter. The angle from the radius OA to the horizontal axis is 2α or twice the angle from the x-axis of the area in question to the axis of maximum moment of inertia.

The angle on the diagram and the angle on the area are both measured in the same sense as shown. The coordinates of any point C are ($I`_x$, $I_{x`y`}$), and those of the corresponding point D are ($I`_y$, $-I_{xy}$). Also the angle between OA and OC is 2θ or twice the angle from the x-axis to the x`-axis. Again we measure both angles in the same sense as shown. We may verify from the trigonometry of the circle that Equations (*), (**), and (***) agree with the statements made.

Products of Inertia and Rotation of Axes...

Example 1

Determine the product of inertia of the rectangular area with centroid at C with respect to the x-y axes parallel to its sides.



Solution.

Since the product of inertia $\bar{I}_{xy}$ about the axes x_0-y_0 is zero by symmetry, the transfer-of-axis theorem gives us

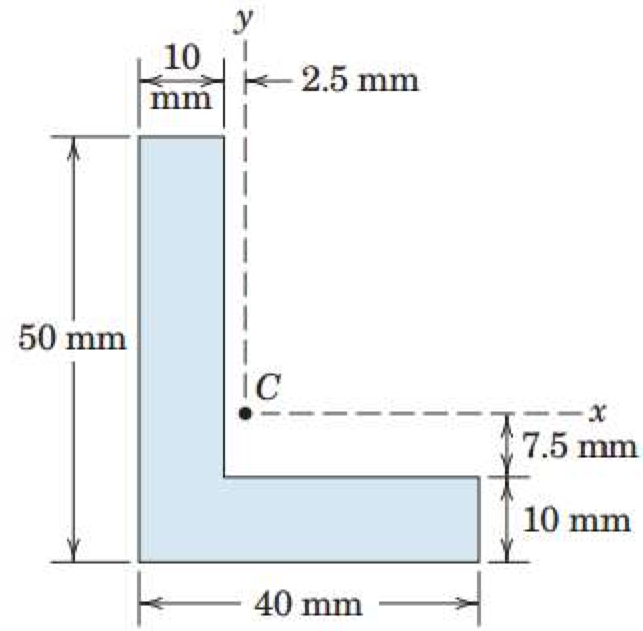
$$[I_{xy} = \bar{I}_{xy} + d_x d_y A] \qquad I_{xy} = d_x d_y bh$$

In this example both dx and dy are shown positive. We must be careful to be consistent with the positive directions of dx and dy as defined, so that their proper signs are observed

Products of Inertia and Rotation of Axes…

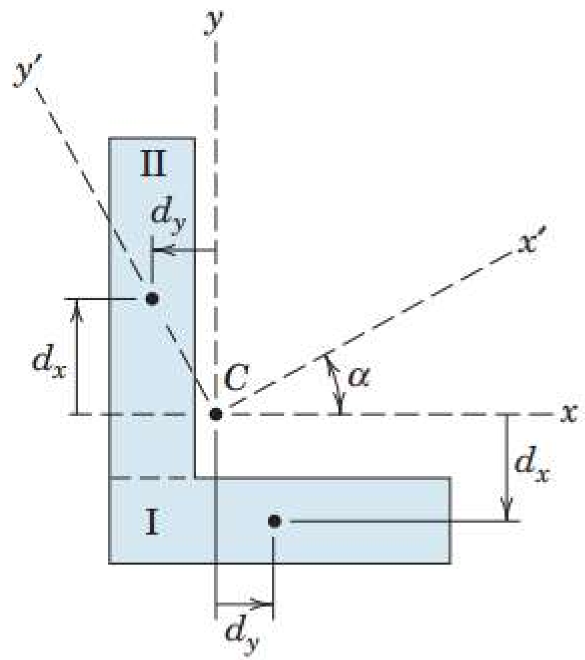
Example 2

Determine the orientation of the principal axes of inertia through the centroid of the angle section and determine the corresponding maximum and minimum moments of inertia.



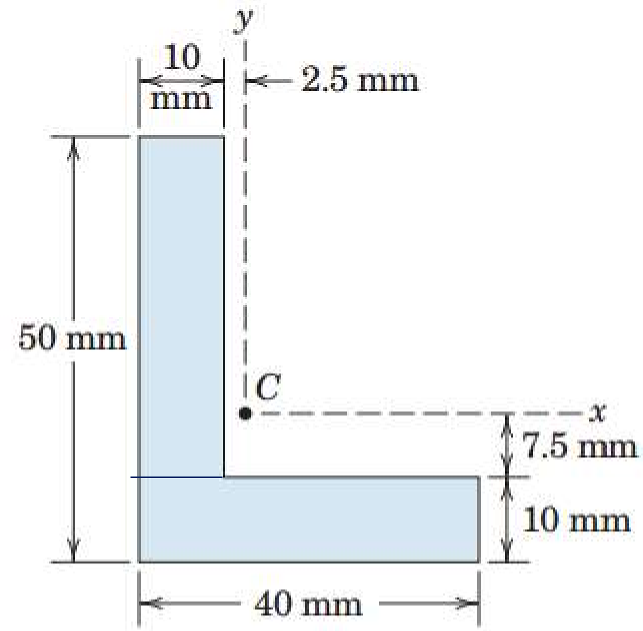
Solution:-

The location of the centroid *C* is easily calculated, and its position is shown on the diagram.



Products of Inertia and Rotation of Axes…

Example 2. Solution…



Product of inertia

For each rectangle about its centroidal axes parallel to the x-y axes is zero by symmetry. Thus, the product of inertia about the x-y axes for part I is

$$[I_{xy} = \bar{I}_{xy} + d_x d_y A]$$

$$d_x = -(7.5 + 5) = -12.5 \text{ mm}$$

$$d_y = +(20 - 10 - 2.5) = 7.5 \text{ mm}$$

$$I_{xy} = 0 + (-12.5)(+7.5)(400) = -3.75(10^4) \text{ mm}^4$$

Likewise for part II, $d_x = +(20 - 7.5) = 12.5 \text{ mm},$

$$d_y = -(5 + 2.5) = -7.5 \text{ mm}$$

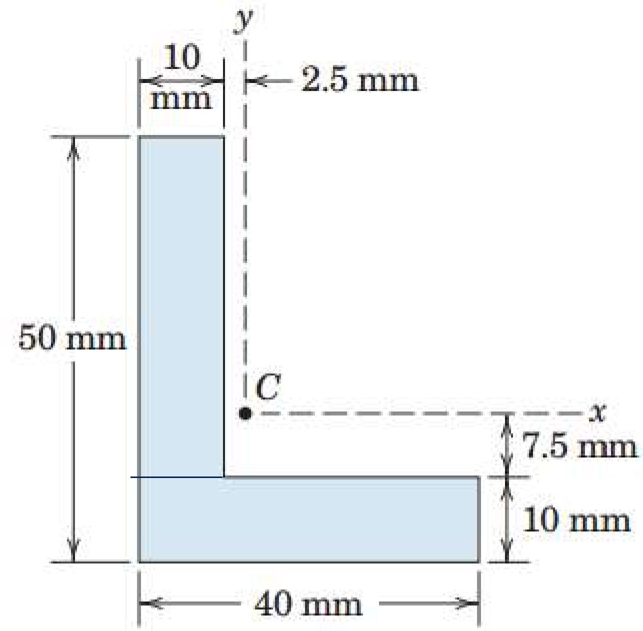
$$I_{xy} = 0 + (12.5)(-7.5)(400) = -3.75(10^4) \text{ mm}^4$$

For the whole angle area $I_{xy} = -3.75(10^4) - 3.75(10^4) = -7.5(10^4) \text{ mm}^4$

Mubarek Z

Products of Inertia and Rotation of Axes…

Example 2. Solution…



Moments of Inertia

The moments of inertia about the x- and y-axes for part I are

$$[I = \bar{I} + Ad^2]$$

$$I_x = \frac{1}{12}(40)(10)^3 + (400)(12.5)^2 = 6.58(10^4)\text{ mm}^4$$

$$I_y = \frac{1}{12}(10)(40)^3 + (400)(7.5)^2 = 7.58(10^4)\text{ mm}^4$$

and the moments of inertia for part II about these same axes are

$$I_x = \frac{1}{12}(10)(40)^3 + (400)(12.5)^2 = 11.58(10^4)\text{ mm}^4$$

$$I_y = \frac{1}{12}(40)(10)^3 + (400)(7.5)^2 = 2.58(10^4)\text{ mm}^4$$

Thus, for the entire section we have

$$I_x = 6.58(10)^4 + 11.58(10)^4 = 18.17(10^4)\text{ mm}^4$$

$$I_y = 7.58(10^4) + 2.58(10^4) = 10.17(10^4)\text{ mm}^4$$

Products of Inertia and Rotation of Axes…

Example 2. Solution…

Principal axes

The inclination of the principal axes of inertia can be calculated by equation(***)

$$\left[\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}\right]$$

$$\tan 2\alpha = \frac{2(-7.50)}{10.17 - 18.17} = 1.875$$

$$2\alpha = 61.9^\circ$$

$$\alpha = 31.0^\circ$$

We now compute the principal moments of inertia from Equation(*) using α for θ and get I_{max} from $I_{x`}$ and $I_{min}$ from $I_{y`}$. Thus,

$$I_{\max} = \left[\frac{18.17 + 10.17}{2} + \frac{18.17 - 10.17}{2}(0.471) + (7.50)(0.882)\right](10^4)$$

$$= 22.7(10^4)\ \text{mm}^4$$

$$I_{\min} = \left[\frac{18.17 + 10.17}{2} - \frac{18.17 - 10.17}{2}(0.471) - (7.50)(0.882)\right](10^4)$$

$$= 5.67(10^4)\ \text{mm}^4$$

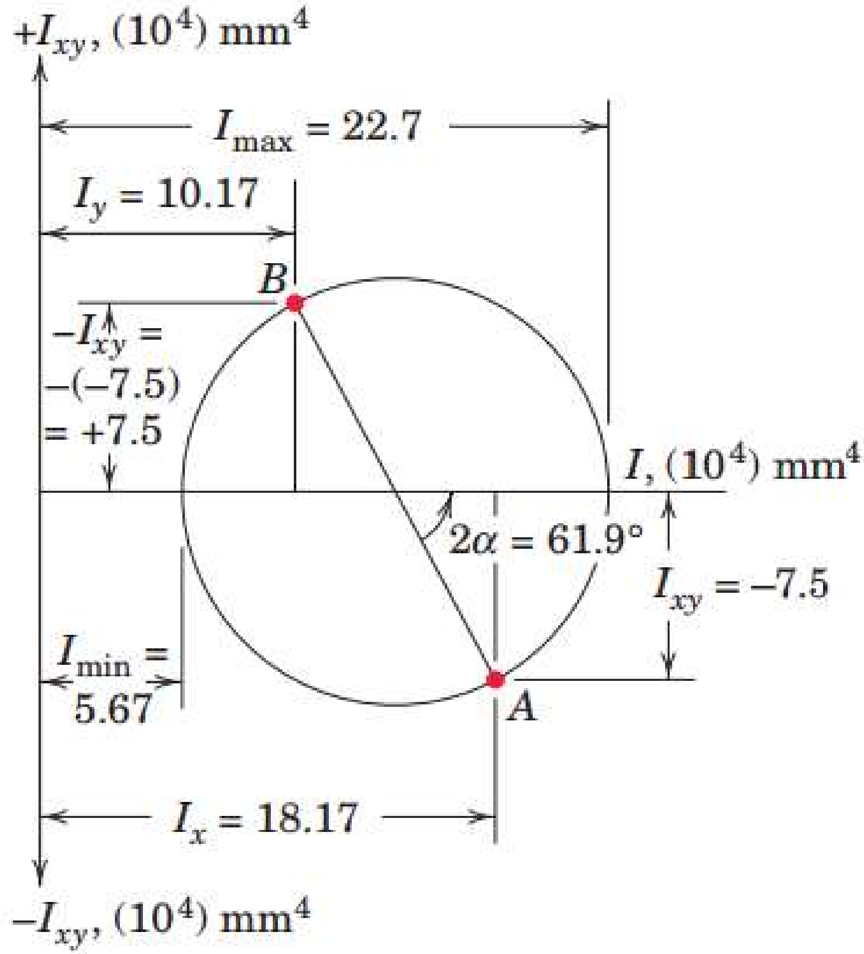
Products of Inertia and Rotation of Axes…

Example 2. Solution…

Mohr's circle

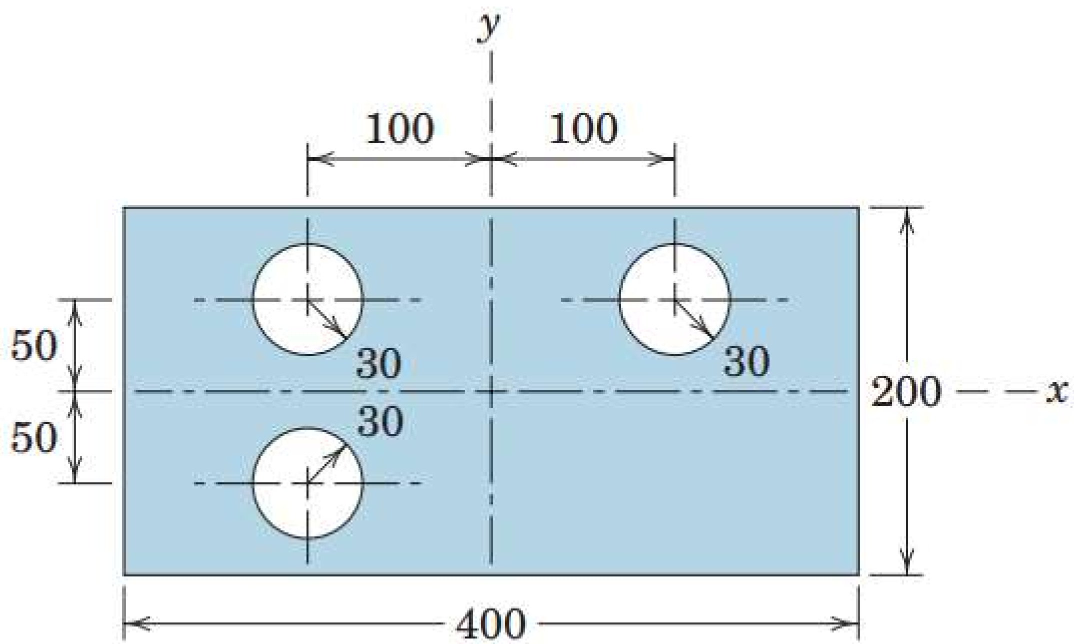
Alternatively to obtain the results for I_{max} and I_{min}, or we could construct the Mohr's circle from the calculated values of I_x, I_y, and I_{xy}. These values are spotted on the diagram to locate points A and B, which are the extremities of the diameter of the circle. The circle is drawn as shown

The angle 2α and I_{max} and I_{min} are obtained from the figure, as shown



Exercise

1. Determine I_x, I_y ,and I_{xy} for the rectangular plate with three equal circular holes. Draw Mohr`s circle



Dimensions in millimeters

Introduction

- **In the preceding chapters we have usually assumed that the forces of action and reaction between contacting surfaces act normal to the surfaces. This assumption characterizes the interaction between smooth surfaces.**
- **Although this ideal assumption often involves only a relatively small error, there are many problems in which we must consider the ability of contacting surfaces to support tangential as well as normal forces.**
- **Tangential forces generated between contacting surfaces are called friction forces and occur to some degree in the interaction between all real surfaces. Whenever a tendency exists for one contacting surface to slide along another surface, the friction forces developed are always in a direction to oppose this tendency.**
- ***In some types of machines and processes we want to minimize the retarding effect of friction forces. Examples are bearings of all types, power screws, gears, the flow of fluids in pipes, and the propulsion of aircraft and missiles through the atmosphere.***
- ***In other situations we wish to maximize the effects of friction, as in brakes, clutches, belt drives, and wedges. Wheeled vehicles depend on friction for both starting and stopping, and ordinary walking depends on friction between the shoe and the ground.***

Introduction…

- Friction forces are present throughout nature and exist in all machines no matter how accurately constructed or carefully lubricated. A machine or process in which friction is small enough to be neglected is said to be ideal. When friction must be taken into account, the machine or process is termed real.
- In all cases where there is sliding motion between parts,
  - the friction forces result in a loss of energy which is dissipated in the form of heat.
  - Wear is another effect of friction.

Types of Friction

(a) Dry Friction

Dry friction occurs when the unlubricated surfaces of two solids are in contact under a condition of sliding or a tendency to slide. A friction force tangent to the surfaces of contact occurs both during the interval leading up to impending slippage and while slippage takes place. The direction of this friction force always opposes the motion or impending motion. This type of friction is also called Coulomb friction. The principles of dry or Coulomb friction were developed largely from the experiments of Coulomb in 1781 and from the work of Morin from 1831 to 1834. we describe an analytical model sufficient to handle the vast majority of problems involving dry friction.

(b) Fluid Friction.

Fluid friction occurs when adjacent layers in a fluid (liquid or gas) are moving at different velocities. This motion causes frictional forces between fluid elements, and these forces depend on the relative velocity between layers. When there is no relative velocity, there is no fluid friction. Fluid friction depends not only on the velocity gradients within the fluid but also on the viscosity of the fluid, which is a measure of its resistance to shearing action between fluid layers. Fluid friction is treated in the study of fluid mechanics and will not be discussed further in this course.

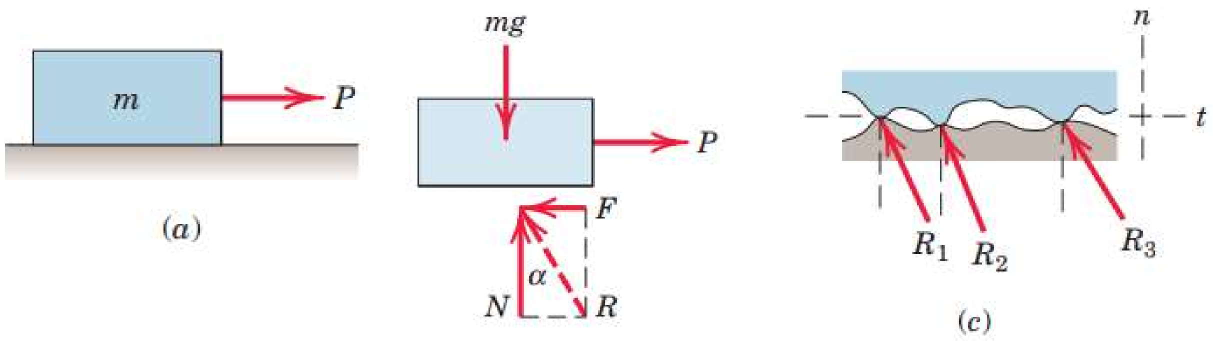
(c) Internal Friction.

Internal friction occurs in all solid materials which are subjected to cyclical loading. For highly elastic materials the recovery from deformation occurs with very little loss of energy due to internal friction. For materials which have low limits of elasticity and which undergo appreciable plastic deformation during loading, a considerable amount of internal friction may accompany this deformation. The mechanism of internal friction is associated with the action of shear deformation, which is discussed in references on materials science.

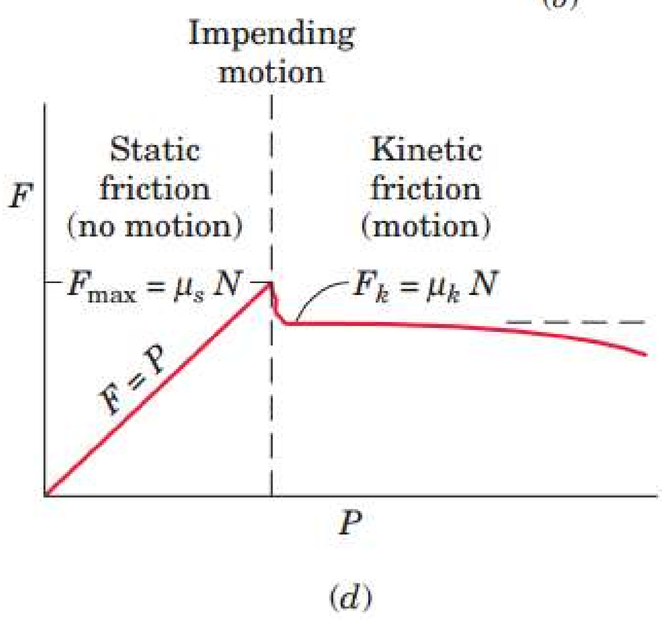
Types of Friction…

(a) Dry Friction

This session describes the effects of dry friction acting on the exterior surfaces of rigid bodies



A magnified view of the irregularities of the mating surfaces, Fig.(c), helps us to visualize the mechanical action of friction.



When the surfaces are in relative motion, the contacts are more nearly along the tops of the humps, and the t-components of the R's are smaller than when the surfaces are at rest relative to one another. This observation helps to explain the well known fact that the force P necessary to maintain motion is generally less than that required to start the block when the irregularities are more nearly in mesh. If we perform the experiment and record the friction force F as a function of P, we obtain the relation shown in Fig. (d). When P is zero, equilibrium requires that there be no friction force. As P is increased, the friction force must be equal and opposite to P as long as the block does not slip. During this period the block is in equilibrium, and all forces acting on the block must satisfy the equilibrium equations. Finally, we reach a value of P which causes the block to slip and to move in the direction of the applied force. At this same time the friction force decreases slightly and abruptly. It then remains essentially constant for a time but then decreases still more as the velocity increases.

Types of Friction…

(a) Dry Friction

Static Friction

The region in Fig. (d) above, up to the point of slippage or impending motion is called the range of static friction, and in this range the value of the friction force is determined by the equations of equilibrium. This friction force may have any value from zero up to and including the maximum value. For a given pair of mating surfaces the experiment shows that this maximum value of static friction F_{max} is proportional to the normal force N. Thus, we may write $F_{\max} = \mu_s N$

where μ_s is the proportionality constant, called the coefficient of static friction.

The equation shown above describes only the limiting or maximum value of the static friction force and not any lesser value. For a condition of static equilibrium when motion is *not* impending, the static friction force is $F < \mu_s N$

Kinetic Friction

After slippage occurs, a condition of kinetic friction accompanies the ensuing motion. Kinetic friction force is usually somewhat less than the maximum static friction force. The kinetic friction force F_k is also proportional to the normal force. Thus, $F_k = \mu_k N$ **; Where μ_k is the coefficient of kinetic friction.**

It follows that μ_k is generally less than μ_s . As the velocity of the block increases, the kinetic friction decreases somewhat, and at high velocities, this decrease may be significant.

Factors affecting friction

- ✓ *Contacting surfaces*
- ✓ *Applied forces*
- ✓ *coefficients of friction*

Which in turn depend greatly on the exact condition of the surfaces, as well as on the relative velocity

Types of Friction Problems

1) The condition of impending motion is known to exist. Here a body which is in equilibrium is on the verge of slipping, and the friction force equals the limiting static friction $F_{max} = \mu_s N$. The equations of equilibrium will, of course, also hold.

2) the condition of neither impending motion nor the condition of motion is known to exist. To determine the actual friction conditions, we first assume static equilibrium and then solve for the friction force F necessary for equilibrium. Three outcomes are possible:

$$(a)\ F < (F_{\max} = \mu_s N) \qquad (b)\ F = (F_{\max} = \mu_s N) \qquad (c)\ F > (F_{\max} = \mu_s N)$$

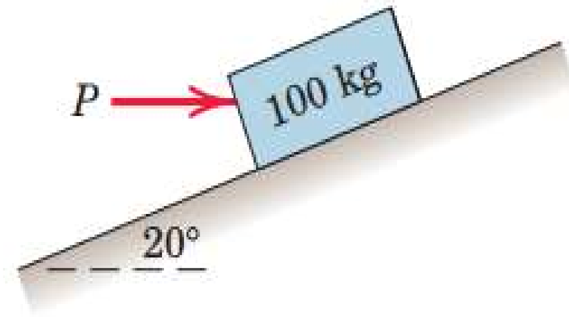
Frist assumption is made and then friction force(F) is calculated and compared with the maximum value (F_{max}). If outcome (c) is happened, then the assumption of equilibrium is therefore invalid, and motion occurs. Thus it will be considered as type 3.

3. Relative motion is known to exist between the contacting surfaces, and thus the kinetic coefficient of friction clearly applies.

The friction force F is equal to $\mu_k N$

Example 1

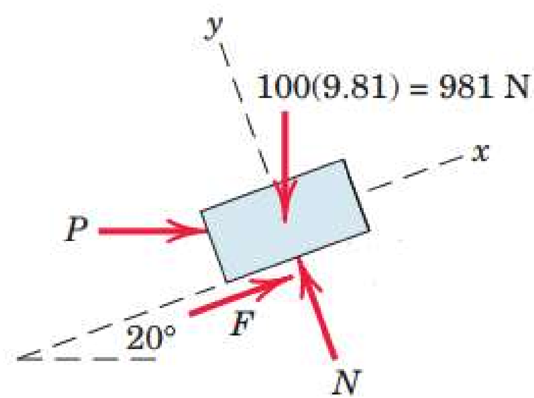
Determine the magnitude and direction of the friction force acting on the 100-kg block shown if, first, P = 500 N and, second, P = 100 N. The coefficient of static friction is 0.20, and the coefficient of kinetic friction is 0.17. The forces are applied with the block initially at rest



Solution: There is no way of telling from the statement of the problem whether the block will remain in equilibrium or whether it will begin to slip following the application of P. It is therefore necessary that we make an assumption, so we will take the friction force to be up the plane, as shown by the solid arrow. From the free-body diagram a balance of forces in both x- and y-directions gives

$$[\Sigma F_x = 0] \quad P\cos 20^\circ + F - 981\sin 20^\circ = 0$$

$$[\Sigma F_y = 0] \quad N - P\sin 20^\circ - 981\cos 20^\circ = 0$$



Case I. P = 500 N

Substitution into the first of the two equations gives

$$F = -134.3 \text{ N}$$

The negative sign tells us that if the block is in equilibrium, the friction force acting on it is in the direction opposite to that assumed and therefore is down the plane. But is the surface capable of developing such force? See next slide

Example 1

Solution…*Case I P=500 …*

To verify that the surfaces are capable of supporting 134.3 N of friction force. This may be done by substituting *P* = 500 N into the second equation, which gives $N = 1093 \text{ N}$

The maximum static friction force which the surfaces can support is then

$$[F_{\max} = \mu_s N] \qquad F_{\max} = 0.20(1093) = 219 \text{ N}$$

Since this force is greater than that required for equilibrium, we conclude that the assumption of equilibrium was correct. The answer is, then,

$$F = 134.3 \text{ N down the plane}$$

Case II. P = 100 N

Substitution into the two equilibrium equations gives $F = 242 \text{ N} \qquad N = 956 \text{ N}$

But the maximum possible static friction force is

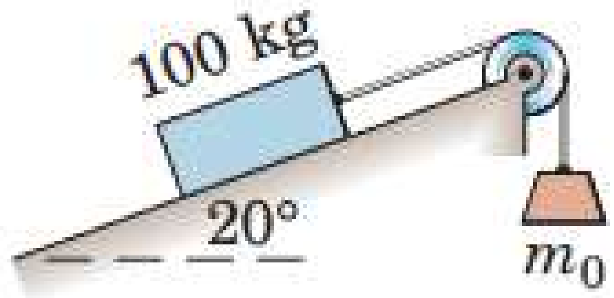
$$[F_{\max} = \mu_s N] \quad F_{\max} = 0.20(956) = 191.2 \text{ N}$$

It follows that 242 N of friction cannot be supported. Therefore, equilibrium cannot exist, and we obtain the correct value of the friction force by using the kinetic coefficient of friction accompanying the motion down the plane. Hence, the answer is

$$[F_k = \mu_k N] \qquad F = 0.17(956) = 162.5 \text{ N up the plane}$$

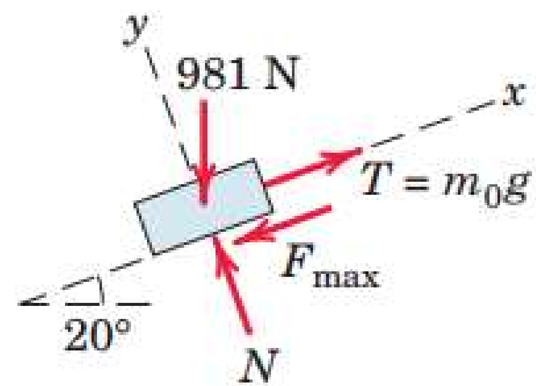
Example 2

Determine the range of values which the mass m_0 may have so that the 100-kg block shown in the figure will neither start moving up the plane nor slip down the plane. The coefficient of static friction for the contact surfaces is 0.30.



Solution.

Case I: The maximum value of m_0 will be given by the requirement for motion impending up the plane. The friction force on the block therefore acts down the plane, as shown in the free-body diagram of the block for Case I in the figure.



Case I

With the weight mg =100(9.81) = 981 N, the equations of equilibrium give

$$[\Sigma F_y = 0] \quad N - 981 \cos 20° = 0 \qquad N = 922 \text{ N}$$

since $$[F_{max} = \mu_s N] \quad F_{max} = 0.30(922) = 277 \text{ N}$$

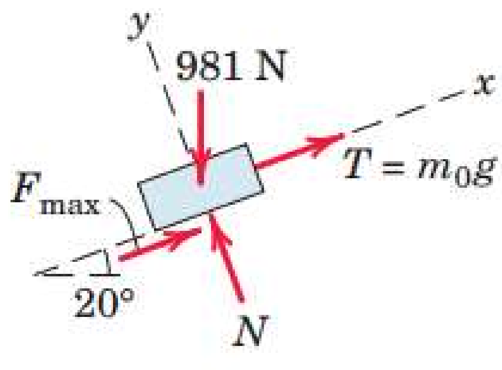
$$[\Sigma F_x = 0] \quad m_0(9.81) - 277 - 981 \sin 20° = 0 \qquad m_0 = 62.4 \text{ kg}$$

 The minimum mass of suspended block ,m_o=62.4 kg to pull up the 100 kg block

Example 2

Solution….

Case II: The minimum value of m_0 is determined when motion is impending down the plane. The friction force on the block will act up the plane to oppose the tendency to move, as shown in the free-body diagram for Case II. Equilibrium in the *x*-direction requires



Case II

$$[\Sigma F_x = 0]$$

$$m_0(9.81) + 277 - 981 \sin 20° = 0 \qquad m_0 = 6.01 \text{ kg}$$

Thus, m_0 may have any value from 6.01 to 62.4 kg, and the block will remain at rest. In both cases equilibrium requires that the resultant of F_{max} and N be concurrent with the 981-N weight and the tension T.

Reading Assignment on problems involving

- **Wedges**
- **Screws**

THE END

Thank You